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Lagrangean Relaxation Approach for Interactive Multiobjective Integer Programming Problem

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〈Abstract〉

Marcott and Soland[3] devleop a method for finding the best solution using a branch and bound method. They used the ideal solution at each stage as means of bounding technique. In this paper we show that that the Lagrangean relaxation idea for scalar optimization problem can be extended to the multiobjective integer programming problem and incoporates an interactive branch and bound method based on Marcott and Soland's method.

Lagrangean Relaxation을 이용한 다수의 목표를 갖는 정수계획법의 해법에 관한 연구

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〈요 약〉

Marcotte와 Soland는 분기한계법(Branch and Bound Method)을 이용한 다수의 목표를 갖는 정수계획법의 해법을 제시하였는데 그 논문에서 Marcott와 Soland는 한계전략(Bounding Strategy)으로서 이상해(Ideal Solution)을 이용하였다.

본 논문에서는 단일목표를 갖는 최적화이론에서 많이 이용되는 Largrangean Relaxation이 다수의 목표를 갖는 정수계획법의 분기한계법에 한계전략으로 이용할 수 있는 것을 보여주고 이를 이용하여 Marcott와 Soland의 해법에 기초를 둔 새로운 해법을 제시하였다.

Ⅰ. Introduction

Marcott and Soland develop a method for finding the best solution at each stage as means of bounding technique. In this paper we show that the Lagrangean relaxation idea for scalar optimization problem can be extended to the multiobjective integer programming problem and incorporates an interactive branch and bound method based on Marcott and Soland's approach.

The essential idea of using Lagrangean method is to provide vector bounds for the objective fuction vector to assist in the elimination of non-optimal solutions. In the next section we will briefly specify the formal assumptions which we will then present the algorithm based on the Marcott and Soland's approach and then illustiated its use. It should be stressed that this is merely an illustration, and as with other approaches to such problems,

a proper numerical evaluation will need to be made in comparison with such methods.

Ⅱ. Assumptions and Underlying theory

The standard scalar discrete optimization problem may be put in the form

maximize $f(x)$

subject to $x \in X$

where $X=\{x \in Z: g_k(x) \leqslant 0, k=1,2,\cdots,p\}$, $Z \subseteq R^n$ is a set of points with integer components and $\{g_k(x)\}$ are valued fuction on Z.

Much standard theory for the scalar problem makes use of the Lagrangean

$$g_\lambda(x)=f(x)-\lambda g(x)$$

where $\lambda \in \Lambda=\{\lambda \in R^p: \lambda \geqslant 0\}$. (see Geoffrin[3] and Shapiro[7])

It is natural to see whether these results may be extended to the vector optimization problem. This takes the form of finding $E(x,f)$ where

$f=\{f^1,f^2,\cdots\cdots,f^m\}: R^n \longrightarrow R^m$

$X=\{x \in Z: g_k(x) \leqslant 0, \ k=1,2,\cdots,p\}$

$E(x,f)=\{x \in X: f^i(y) \geqslant f^i(x)$ for some $y \in X, \ i=1,2,\cdots,m$ imply $f^i(y)=f^i(x), \ i=1,2,\cdots,m\}$.

The set $E(X,f)$ is the vector maximal set and is the gloval pareto maximal set.

For this problem the simplest natural generalization of the scalar Lagrangean is the Lagrangean

$$g_\lambda(x)=f(x)-g(x)e$$

where, again $\lambda \in \Lambda$, and $e \in R^m$ has unit components.

The vector Lagrangean has been studied by various other arthurs, the main, and more recent ones, being Sarawagi and Tanino[6], Bitran[1], Brumelle[2], Kawasaki[4].

Let assume that the decision maker has an implicit value fuction u on R which is monotonic increasing on R. It is required to find a point in X which maximizes $u(f(x))$, but u is assumed not to be known explicitly.

Beacause of the monotonicity assumption we would only need confine our search to $E(X,f)$. If we were able to determine the whole of $E(X,f)$, then this would be the end of the matter. In general this is not the appropriate thing to do and one might seek an appropriate subset of $E(X,f)$. Even here this may create difficulties. and we may seek some surrogate representation which is still effective enough in its resolution of the initial problem. Such a surrogate may be $E(Z,g_\lambda)$, and we will use this in the suggested algorithm. The significance of $E(X,f)$ is the following natural extension of the corresponding scalar result, which we state without proof since this trivial

$$x \in X, \ \lambda \in \Lambda, \ z \in E(X,f) \longrightarrow f(x) \not\geqq g_\lambda(z)$$

where $\geq$means$\geqq$but not$=$in a vector sense.

In seeking $E(X,f)$ we may make use of this for a bounding and elimination process. In the scalar case, for the above circumstances we have $f(x) \not\geqq g_\lambda(z)$ and this is then placed as a constraint on the next stage of the computation. A similar device is used in the algorithm of this paper.

It is clear that, for a given z, if would be useful to choose λ so that $g_\lambda(z)$ is as good as is possible in its elimination capabilities. We proceed as follows.

Let $z \in Z$. Define

$$\Lambda(z)=\{\lambda \in \Lambda: z \in E(X,g_\lambda)\}$$

Then

$$x \in X \longrightarrow f(x) \not\geqq g(z) \text{ for any } \lambda \in \Lambda.$$

Now

$$g_\lambda(z)=f(z)-\lambda g(z)e.$$

Let us suppose that $\max_{\lambda \in \Lambda(z)} \lambda g(z)$ exists and takes its value at $\lambda(z) \in \Lambda(z)$. Then

$$x \in X \longrightarrow f(x) \not\geqq g_{\lambda(z)}(z).$$

If $\Lambda'(z) \subseteq \Lambda(z)$, and with the obvious definittion of $\lambda'(z)$, we also obtain, assuming a maximzer exists,

$$x \in X \longrightarrow f(x) \not\geqq g_{\lambda'(z)}(z).$$

since

$$g_{\lambda'(z)}(z) \geqq g_{\lambda(z)}(z).$$

These bounds are useful in the interactive branch and bound section later on. They need only be invoked if a particular subset of X from further consideration. $A'(z)$ is a surrogate for $A(z)$ if $A(z)$ is difficult to determine.

It is not always true that (z) is closed. This applies in the illustration given later on. In such cases

$$\max_{\lambda \in A(z)} \lambda g(z)$$

may not exist. In such cases we may take

$$\sup_{\lambda \in A(z)} \lambda g(z),$$

or alternatively

$$\max_{\lambda \in A(z)} \lambda g(z)$$

since this will exist, for otherwise we can make $\lambda g(z)$ arbitrarily large, and this would imply $X=\phi$.

As will be seen, the method given later on does not depend on

$$\max_{\lambda \in A(z)} g(z)$$

existing.

Let us now consider the interactive algorithm.

Ⅲ. An interactive branch and bound method

Marcotte and Soland develop a method for finding the best solution in a set X using a branch and bound method.

It is assumed that a value fuction, monotone increasing on $f(X)$, exists and that, hence, we may restrict ourselves to $E(X^k, f)$ at each stage, where X^k is any one of subsets of X which the method generates.

A typical step is as follows, where at some stage we are restricted to examining a set of sets $\{X^k\}$, $X^k \subset X$, and where it is known that some optimum lies in some X^k. At this stage x^* is a best point to date obtained from comparisons of all specific points identified to date.

Consider X^k. Let σ^k be the ideal point of X^k. If, by interaction with the decision maker, x^* is preferred, or indifferent, to σ^k we may not eliminate X^k. Otherwise we select a point $x^k \in E(X^k, f)$. Since we are restricting ourselves to $E(X, f)$, X^k is then split into m, not necessarily disjoint, subsets $\{X^{ki}\}$ where

$$X^{ki}=\{x \in X^k : f^i(x) > f^i(x^k)\},$$
$$i=1, 2, \cdots\cdots, m,$$
$$f(E(X^k, f)) \subseteq U_i f(X^{ki}).$$

This is done for all $\{X^k\}$, and a best point to date becomes a best point of $\{x^*, x^1, x^2 \cdots\cdots, x^k\}$. The procedure is repeated for all the $\{X^k\}$ which have not been eliminated. Exentually a bett point in X will be found.

Using the Lagrangean method it is possible to introduce Lagrangean multipliers for each constraint introduced at each stage. This will quickly make the procedure unmanageable in general, and hence, we will not do this. On the other hand, at this stage we will have subsets $\{X^k\}$ of X and associated subsets $\{Z^k\}$ of Z, and it is important that the determination of $E(Z^k, g_\lambda)$ be simpler than the determination of $E(X^k, f)$. This will not be the case with the branching method of Marcotte and Soland in terms of inequalities on $\{f^i\}$. We will keep Z^k in a similar rectangular form to Z, and hence, will branch by splitting the range of z_j, for some j, at each stage. We now describe a typical step.

Suppose, like Marcott and Soland, we have reached a stage where we know that an optimal solution lies in one of a set of subsets $\{X^k\}$ of X and we have a best point $x^* \in X$ to date from among the specific points we have considered. Let us take a typical X and let it take the form

$$X^k=\{x \in X : x_j \in [l_{jk}, n_{jk}],\ j=1, 2, \cdots\cdots, n\}$$

where (l_{jk}, u_{kj}) are non-negative integers, and where we allow $u_{jk}=\infty$ when x_j is unbounded.

We set

$Z^k=\{z\in Z:\ z_j\in[l_{jk},\ u_{jk}],\ j=1,2,\cdots,n\}$.

Select a points $x^k\in X^k$, a point $\lambda\in\Lambda$ and a point $z^k\in E(Z^k,\ g_\lambda)$. Determine $g_\lambda(z^k)$ and the ideal objective fuction vector $f(\sigma^k)$ of X^k (this is not necessary, but in some cases we have $g_\lambda^i(z^k)>f^i(\sigma^k)$ for some i, and the ideal point is better for elimination purpose). Set, for $i=1,2,\cdots,m$,

$h_\lambda^i(z^k)=\min[f^i(\sigma^k),\ g^i(z^k)]$,

$q_{i\lambda}(z^k)=(q_{i\lambda}^s(z^k))\in R^m,\ s=1,2,\cdots,m$,

$q_{i\lambda}^i(z^k)=h_\lambda^i(z^k),\ q_{i\lambda}^s(z^k)=f^s(\sigma^k),\ \forall s\neq i$.

Then it is clear that

$X^k\subseteq_i Y^{ki}$

where

$Y^{ki}=\{x\in X^k:\ f(x)\leqq q_{i\lambda}(z^k)\},\ i=1,2,\cdots\cdots,m$.

If $f(x^*)$ is better than or indifferent to $q_{i\lambda}(z^k)$, $\forall i$, then we may eliminate X^k. If $f(x^*)$ is worse than $q_{i\lambda}(z^k)$, for some i then we may try to improve the bound $g_\lambda(z^k)$ as indicated in Section 2.

We do this for all the $\{X^k\}$ subsets. For each subset not eliminated we branch by subsiding the range of x_j for some j to ensure that Z^k remains rectangular as each stage. We update x^* by taking the best of the previous x^* and the sample points $\{x^k\}$. The procedure is repeated until an optimal solution is found.

At each stage we may choose several $\{x^k,\ \lambda,\ z^k\}$ combinations to improve the prospects of elimination subsets.

Also the method is easily modified to ensure that all optima are obtained as distinct from a single optimum as the method stands.

We will now consider a simple linear example, merely to illustrate the procedure. The initial calculations illustrate the calculation of $E(Z, g_\lambda)$ for a selection of λ values.

V. An example

Let

$X=\{x\in R^2:\ \frac{1}{2}x_1+\frac{1}{4}x_2\leqslant 8,\ \frac{1}{5}x_1+\frac{1}{5}x_2\leqslant 4\}$

$f^1(x)=x_1+3x_2,\ f^2(x)=3x_1-2x_2$,

$g_1(x)=\frac{1}{2}x_1+\frac{1}{4}x_2-8,\ g_2(x)=\frac{1}{5}x_1+\frac{1}{5}x-4$,

$Z=\{z\in R_+^2:\ z\geqslant 0\},\ \Lambda=\{\lambda\in R^2:\ \lambda\geqq 0\}$.

Then it is easily seen that

$g(z)=z-\lambda_1 g_1(z)-\lambda_2 g_2(z)$

$=h\lambda(z)+c(\lambda)$

$c(\lambda)=(18\lambda_1+14\lambda_2)e$

$h\lambda(z)=p_\lambda z_1+q_\lambda z_2$

$p_\lambda=\begin{pmatrix}1-\frac{1}{2}\lambda_1-\frac{1}{5}\lambda_2\\ 3-\frac{1}{2}\lambda_1-\frac{1}{5}\lambda_2\end{pmatrix}z_1$

$q_\lambda=\begin{pmatrix}3-\frac{1}{4}\lambda_1-\frac{1}{5}\lambda_2\\ 2-\frac{1}{4}\lambda_1-\frac{1}{5}\lambda_2\end{pmatrix}z_2$

Let us assume that a true value function exist of the form $u(f)=2f^1-f^2=5x_1+8x_2$

Select some point $x^*\in X$, e.g. $x^*=(10, 10)$

Set $X^1=\{x\in X:\ x\leqq 5\}$, $X^2=\{x\in X:\ x\geqq 5\}$ and $Z^1=\{z\in Z:\ z\leqq 5\}$, $Z^2=\{z\in Z:\ z\geqq 5\}$

Select $\lambda\in\Lambda$, e.g. $\lambda=(10/3,\ 0)$,

$z^1\in E(Z^1,\ g_\lambda)=(0,\ 20)$

$g_\lambda(z^1)=\begin{pmatrix}\left(3-\frac{1}{4}\times\frac{20}{6}-0\right)\times 20+\left(8\times\frac{20}{6}+0\right)\\ \left(2-\frac{1}{4}\times\frac{20}{6}-0\right)\times 20+\left(8\times\frac{20}{6}+0\right)\end{pmatrix}=\begin{pmatrix}70\\ 50\end{pmatrix}$

Consider the ideal point of X^1, $\sigma^1=\left(5, 18\frac{1}{3}\right)$

$f^1(\sigma^1)=60,\ f^2(\sigma^2)=51\frac{2}{3}$

Hence

$h_\lambda^1(z^1)=\min[f^1(\sigma^1),\ g_\lambda^1(z^1)]$

$=\min[70,\ 60]=60$

$h_\lambda^2(z^2)=\min[f^2(\sigma^1),\ g_\lambda^2(z^1)]$

$=\min\left[50,\ 51\frac{2}{3}\right]=50$

$q_{i\lambda}(z^1)=\begin{pmatrix}60\\ 51\frac{2}{3}\end{pmatrix}=y$ or $\begin{pmatrix}70\\ 50\end{pmatrix}=z$

$u(y)=2\times 60+51\frac{2}{3}=171\frac{2}{3}>130=u(x^*)$

$u(z)=2\times 70+50=190>130=u(x^*)$

Therefore X^1 is not eliminated and we update x^* by taking the best of the previous x and the sample ponts $\{x^k\}$, i.e. $x^*=(0, 20)$.

Now we consider X^2. Select the some point $z^2\in E(z^2, g_\lambda(z))$, e.g. $z=(12, 8)$ and $\lambda\in\Lambda$, e.g. $\lambda\in(20/6, 0)$.

We have

$$g_\lambda(z_2)=\begin{pmatrix}36\\52\end{pmatrix}-\frac{20}{6}=\begin{pmatrix}32\frac{2}{3}\\48\frac{2}{3}\end{pmatrix}$$

$$\sigma^2=\left(12,\ 12\frac{2}{3}\right),\ f^1(\sigma^2)=12+3\times12\frac{2}{3}=50$$

$$f^2(\sigma^2)=3\times12+2\times12\frac{2}{3}=51\frac{1}{3}$$

$$h^1_\lambda(z^2)=\min[f^1(\sigma^2),\ g^1_\lambda(z^2)]=\min\left[50,\ 32\frac{2}{3}\right]=32\frac{2}{3}$$

$$h^2_\lambda(z^2)=\min[f^2(\sigma^2),\ g^2_\lambda(z^2)]=\min\left[48\frac{2}{3},\ 51\frac{1}{3}\right]=48\frac{2}{3}$$

$$q_{1\lambda}(z^2)=\begin{pmatrix}32\frac{2}{3}\\51\frac{1}{3}\end{pmatrix}=y \text{ or } \begin{pmatrix}50\\48\frac{2}{3}\end{pmatrix}=z$$

Therefore

$$u(y)=2\times32\frac{2}{3}+51\frac{1}{3}=116\frac{2}{3}<160=u(x^*)$$

$$u(z)=2\times50+48\frac{2}{3}=148\frac{2}{3}<160=u(x^*)$$

Hence X^2 is eliminated from further consideration.

Ses $X^3=\{x\in x^1:\ X_2\leqslant15\}$, i.e.

$$x^3=\{x\in X:\ x_1\leqslant5,\ x_2\leqslant15\}$$

$$X^4=x^1\backslash x^3=\{x\in X:\ x_1\leqslant5,\ x_2\geqslant15\}$$

Consider the set X^3. Select the same point $z^3=E(z^3, g_\lambda(z^3))$, e.g. $z^3=(5, 15)$ and $\lambda\in\Lambda$, e.g. $\lambda=(20/6, 0)$

we have

$$g_\lambda(z)=\begin{pmatrix}50\\45\end{pmatrix}+\frac{7}{4}\lambda_1=\begin{pmatrix}55\frac{5}{6}\\55\frac{5}{6}\end{pmatrix}$$

$$\sigma^3=(5,\ 15),\ f^1(\sigma^3)=50,\ f^2(\sigma^3)=45$$

$$h^1_{1\lambda}(z^3)=\min[f^1(\sigma^3),\ g^1_\lambda(z^3)]=50,$$

$$h^2_{1\lambda}(z^3)=\min[f^2(\sigma^3),\ q^2_1(z_6)]=45$$

$$q_{1\lambda}(z^3)=\begin{pmatrix}50\\45\end{pmatrix}=y$$

$$u(y)=2\times100+45=145<160=u(x^*)$$

Hence X^3 is eliminated from further consideration.

Now consider the set X^4. Subdivide X into

$$X^5=\{x\in X^4:\ x_1\leqslant3,\ x_2\geqslant15\}$$

$$X^6=\{x\in X^4:\ 3\leqslant x_1\leqslant5,\ x_2\geqslant15\}$$

and continue by selecting points in X^5, X^6, comparing with x^* to find the best to date, selecting $\lambda\in\lambda$, finding $z^k=E(z^k, g_\lambda)$, $k=5,6$, and so on. Eventually the procedure will terminated at an optimal solution, viz., $x^*=(0, 20)$, $u(f^*)=160$.

In the above special problem it is clear that the choices of $x^k\in X^k$ and $z^k\in E(Z^k, g_\lambda)$ could obviously be made in more efficient ways, but for more general problems this may not be so easy. Hence, to some extents, these points have been chosen arbitarily but at the same time chosen to illustrate particular features of the method.

V. Summary and comments

This paper explores the possibility of extending relaxation technique from the scalar problem to the vector problem. It is seen that analogous method do exist which may be useful particulary when a surrogate set of solution for a subset of the efficient solutions is required, from which the final solution will be chosen, or where an optimal solution is to be obtained by interaction with the decision maker. In the latter case, a branch and bound method is suggested which is stronger than the use of ideal points as suggested by Marcott and Soland.

This paper is merely exploratory and, as with modern developments in the use of Larangean methods for some discrete problems, extensive numerical work on realistic problems is needed to see if the method is practicable and competitive.

The essential advantage rests in the ability to determine numbers of $E(Z, g_\lambda)$ rather more easily than determining members of $E(X, f)$, or even approximately members of the latter.

Clearly there are some open problems associated with the suggested procedure such as:

(a) How can the choice of $\lambda \in \Lambda$, and $z \in E(Z, g_\lambda)$ be sensibly guided?

(b) It is possible to determine etiher a good algorithm to find $\lambda(z)$, or a sufficiently good subset of $\Lambda(z)$?

(c) Would the replacement of $\lambda \in \Lambda$ by $\lambda \in \{\lambda \in R^{m \times k}: \lambda \geqq 0\}$ and $g_\lambda(x)$ by $f(x) - \lambda g(x)$ prove to be a more competitive approach?

(d) Would it be better to introduce extra constraints which arise in the branch and bound method in the Lagrangean from, thus widening the types of branch and bound procedure which might be used?

Such questions would form the basis of future research.

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