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On the Numerical Range for Linear Operators in Hilbert Space

Park, Young-Sik · Lee, Je-Yoon
Dept. of Mathematics
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〈**Abstract**〉

Using the concept of the numerical range $W(T)$ of a linear operator T in Hilbert space H, we show the relations between $W(T)$ and some spectrums of T. Moreover, their properties reduce a non-normal operator to a normal operator. Finally, we investigate the properties of an extreme point of $W(T)$ and some spectrums of linear operators in Hilbert space.

Hilbert space에서의 선형작용소들에 대한 numerical range에 대하여

박영식 · 이제윤
수 학 과
(1985. 9. 30. 접수)

〈요 약〉

numerical range $W(T)$의 개념을 사용하여 $W(T)$와 작용소 T의 여면 spectrum들과의 관계를 보이고 나아가 그들 성질들로 부터 normal인 작용소를 찾아, $W(T)$의 extreme point와 Hilbert space의 선형작용소의 spectrum들의 특성들을 살펴본다.

I. Introduction

In this paper H is a separable, infinite dimensional complex Hilbert space with inner product $(,)$, and the Banach algebra of all bounded linear operators in H will be denoted by $L(H)$. Denoted by $\sigma(T)$ the spectrum, by $\sigma_p(T)$ the point spectrum, by $\sigma_a(T)$ the approximative point spectrum, by $\sigma_r(T)$ the residual spectrum, and by $W(T)=\{\lambda : \lambda=(Tx, x), \|x\|=1\}$ the numerical range of T. As is well known, $W(T)$ is a convex set (by Theorem of Toeplitz and Hausdorff), $\sigma(T)$ is a compact set moreover, $\sigma_p(T)\subset\sigma_a(T)\subset\sigma(T)\subset\overline{W(T)}$, closure of $W(T)$ (by Wintner [11]), $\sigma_p(T)\subset W(T)$ and $\sigma_a(T)\supset\partial\sigma(T)$=boundary of $\sigma(T)$. Furthermore, let $\|T\|=\max_{\|x\|=1}\|Tx\|$, $\omega(T)=\sup\{|\lambda| : \lambda\in W(T)\}$ =numerical radius of T and $r(T)=\sup\{|\lambda| : \lambda\in\sigma(T)\}$=spectral radius of T. Then the inequality $r(T)\leq\omega(T)\leq\|T\|$ holds. We denote by $E(T)$ the set of the extreme points of the compact convex set $\overline{W(T)}$ and by Conv M the convex hull of a set M of the complex plane C. We have Conv $E(T)=\overline{W(T)}$ (because C is two dimensional). W.F. Donghue lists the following most important fact about $W(T)$;

(1) $W(T)$ is convex [6].

(2) $\overline{W(T)}\supset\sigma(T)$ [10].

(3) If T is normal, $\overline{W(T)}$ is the smallest closed convex set containing $\sigma(T)$. [10].

(4) If T is normal and $W(T)$ is closed, the extreme points of $W(T)$ are eigen values [6].

(5) If $W(T)$ reduces to the single point λ, then $T=\lambda I$, where I is the identity [1].

(6) If $W(T)$ is a subset of the real axis, T is self-adjoint [1].

From the above statements, it is the aim of this paper to prove some relations between spectras and the numerical range, which are known in the case of normal operators, for a larger class of operators.

II. Numerical range and some spectrums.

Let $T\in L(H)$ and ζ be a complex number, and let $E_\zeta[T]=\{x\in H : Tx=\zeta x\}$. Of course, $E_\zeta[T]\neq 0$ if and only if $\zeta\in\sigma_p(T)$. It is well known that $E_\zeta[T]=E_{\bar\zeta}[T^*]$ for every normal operator T. Let $A_\zeta[T]=\{\{x_n\}$ in $H : \|x_n\|=1, (Tx_n, x_n)\to\zeta(n\to\infty)\}$, Saitô [7], if a point $\zeta\in\sigma_p(T)$ satisfies the relation $E_\zeta[T]=E_{\bar\zeta}[T^*]$, ζ is called a normal eigen value of T.

Lemma 2.1 Let T be a contraction (i.e., $\|T\|\leq 1$) and $\lambda\in\sigma_p(T)$ of modulus 1, then $E_\lambda[T]=E_{\bar\lambda}[T^*]$ holds.

Proof. Let U be a unitary dilation of T, then $\|Ux-\lambda x\|^2=2\|\lambda x\|^2-2\mathrm{Re}\bar\lambda(Ux, x)\leq 2\|x\|\|U-\lambda I\|x$ for all $x\in H$. Thus we have $Tx_n-\lambda x_n\to 0(n\to 0)$ if and only if $Ux_n-\lambda x_n\to 0(n\to\infty)$ for a sequence $\{x_n\}$ of unit vectors of H. Similarly $T^*x_n-\bar\lambda x_n\to 0(n\to\infty)$ if and only if $U^*x_n-\bar\lambda x_n\to 0(n\to\infty)$ Since U is unitary, $Ux_n-\lambda x_n\to 0(n\to\infty)$ is equivalent to $U^*x_n-\bar\lambda x_n\to 0(n\to 0)$. Thus $Tx=\lambda x$ if and only if $T^*x=\bar\lambda x$. Therefore the relation $E_\lambda[T]=E_{\bar\lambda}[T^*]$ holds.

Theorem 2.2 Let T be a contraction. Then $\lambda\in\overline{W(T)}$ if and only if there is a sequence $\{x_n\}$ of $A_\lambda[T]$ such that $(T^*x_n, x_n)\to\bar\lambda(n\to\infty)$.

Proof. If $\lambda\in\overline{W(T)}$, then there is a sequence $\{x_n\}$ of unit vectors such that $(Tx_n, x_n)\to\lambda(n\to\infty)$ (i.e., $\{x_n\}\in A_\lambda[T]$) Since T is a contraction, T has not only normal but also unitary dilation. By Lemma 2.1, let U be a unitary dilation of T, we have $(Tx_n, x_n)\to\lambda(n\to\infty)$ if and only if $(Ux_n, x_n)\to\lambda(n\to\infty)$ for a sequence $\{x_n\}$ of unit vectors of H. Since U is unitary, $(Ux_n, x_n)\to\lambda(n\to\infty)$ is equivalent to $(U^*x_n, x_n)\to\bar\lambda(n\to\infty)$ if and only if $(T^*x_n, x_n)\to\bar\lambda(n\to\infty)$. Conversely, suppose that there is a sequence $\{x_n\}$ of $A_\lambda[T]$ such that $(T^*x_n, x_n)\to\bar\lambda(n\to\infty)$. Similiarly it follows that $(T^*x_n, x_n)\to\bar\lambda$ if and only if $(Tx_n, x_n)\to\lambda(n\to\infty)$, hence $\lambda\in\overline{W(T)}$.

Corollary 2.3. If T is a contraction and $|\zeta|=1$, we have $A_\zeta[T]=A_{\bar\zeta}[T^*]$, and the relation $A_\zeta[T]=A_{\bar\zeta}[T^*]$ implies that $Tx_n-\zeta x_n\to 0(n\to\infty)$ if and only if $T^*x_n-\bar\zeta x_n\to 0(n\to\infty)$ for a sequence $\{x_n\}$ of unit vectors.

Proof. From Lemma 2.1. and Theorem 2.2.

Corollary 2.4. If $\zeta\in(\sigma_a(T)\cup\sigma_r(T))\cap\partial W(T)$, then $A_\zeta[T]=A_{\bar\zeta}[T^*]$ and $E_\zeta[T]=E_{\bar\zeta}[T^*]$, where $\partial W(T)$ is the boundary of $W(T)$.

Proof. If $\zeta\in\sigma(T)\cap\partial W(T)$ and $\zeta\in\sigma_a(T)$, then there is a sequence$\{x_n\}$ in H with $\|x\|=1$ and $\lim_{n\to\infty}\|(\zeta I-Tx_n)\|=0$.

By translation, we can suppose that $\zeta=0$ and Re $W(T)\geq 0$. Hence there is a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\|\to 0$ $(n\to\infty)$. Since $(Tx_n, x_n)\to 0(n\to\infty)$ we have $((\mathrm{Re}T)x_n, x_n)=\mathrm{Re}(Tx_n, x_n)\to 0$ $(n\to\infty)$. From Re $W(T)\geq 0$, since $Tx_n=(\mathrm{Re}T)x_n-i(\mathrm{Im}T)x_n$, $\|(\mathrm{Im}\ T)x_n\|=\|Tx_n-(\mathrm{Re}\ T)x_n\|\leq\|Tx_n\|+\|(\mathrm{Re}\ T)x_n\|\to 0$ $(n\to\infty)$ Thus $\|T^*x_n\|=\|(\mathrm{Re}\ T)x_n-i\ (\mathrm{Im}\ T)x_n\|\leq\|(\mathrm{Re}\ T)x_n\|+\|(\mathrm{Im}\ T)x_n\|\to 0(n\to\infty)$, and so $A_\zeta[T]\subseteq A_{\bar\zeta}[T^*]$. By a symmetric argument $A_\zeta[T]\supseteq A_{\bar\zeta}[T^*]$, therefore $A_\zeta[T]=A_{\bar\zeta}[T^*]$. Also, the relation $E_\zeta[T]=E_{\bar\zeta}[T^*]$ holds.

Definition 2.5. [5] [7] $T\in L(H)$ is hyponormal if $T^*T-TT^*\geq 0$, which is equivalent to $\|T^*x\|\leq\|Tx\|$ for all $x\in H$, T is normaloid if $\|T\|=r(T)$; equivalently, $\|T\|=\sup\{|Tx, x)| : \|x\|=1\}=\omega(T)$, and T is restriction convexoid if the restriction of T to any invariant subspace is convexoid, which is equivalent to $\overline{W(T)}=\mathrm{Conv}(\sigma(T))$.

Lemma 2.6. [7] Let T be restriction convexoid. If $\sigma(T)$ is finite, then T is normal.

Theorem 2.7. Let T be restriction convexoid

and $\sigma(T)$ be finite, and let $K=\{x\in H : Tx=e^{i\theta}T^*x\}$ for a fixed real value θ. Then K is a reducing subspace of T and $T|K$ is normal, thus $T^n x=e^{in\theta}T^{*n}x$ for $x\in K$.

Proof. It follows that $Tx=e^{i\theta}T^*x$ implies $\|Tx\|=|e^{i\theta}|\|T^*x\|=\|T^*x\|$ for $x\in K$. From Lemma 2.6 T is normal, and so T is hyponormal. Therefore $(T^*T-TT^*)x,\ x)=0$ for all $x\in H$ and for $y\in H$ $|((T^*T-TT^*)x,\ y|^2\leq|((T^*T-TT^*)x,\ x)|\cdot|((T^*T-TT^*)y,\ y)|=0$ by the generalized Schwarz inequality for positive operators. Since y is arbitrary, we have $T^*Tx=TT^*x$. Hence we have $T(T^*x)=T^*(Tx)=T^*(e^{i\theta}T^*x)$ for $x\in K$, and $T(Tx)=T(e^{i\theta}T^*x)=e^{i\theta}T^*(Tx)$ for $x\in K$, thus $T^*K\subset K$ and $TK\subset K$. Therefore K is a reducing subspace of T. Also, since $T^*Tx=TT^*x$ holds for all $x\in K$, $T|K$ is normal. Since $T^2x=T(Tx)=e^{i\theta}(TT^*x)=e^{2i\theta}T^{*2}x$ for all $x\in K$, if $T^{n-1}x=e^{i(n-1)\theta}T^{*(n-1)}x$ holds for all $x\in K$, Then $T^n x=T((T^{n-1}x)=e^{in\theta}T^{*n}x$ holds for all $x\in K$ (by induction).

Lemma 2.8. [3] T is normaloid if and only if $\|T\|=\omega(T)=r(T)$.

Example 2.9. [7] Let T be an operator on a three-dimensional defined by $T=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}$ with respect to the orthonormal basis $\{e_1,\ e_2,\ e_3\}$. Then $\sigma(T)=\{0,\ 1\}$ and $\|T\|=r(T)=1$, so that T is normaloid.

Example 2.10. [3] Let T_1 be an operator on the two-dimensional space H_1 defined by $T_1=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Then $1/2=\omega(T_1)<\|T_1\|=1$, $\sigma(T_1)=\{0\}$.

Let $T_2=(t_{ik})_{i,k=1,2,\cdots,\infty}$ be the matrix with $t_{ii}=\lambda_i$ and $t_{ik}=0$ for $i\neq k$, and T_2 defines a normal bounded operator on an infinite-dimensional spacc H_2 with $\sigma(T_2)=W(T_1)$. Since T_2 is normal, we have $\omega(T_2)=\|T_2\|$. Finally, we take the operator $T=T_1\oplus T_2$ in $H=H_1\oplus H_2$. Then we have $\omega(T)=\omega(T_1)=\omega(T_2)=\|T_2\|$ and hence $\|T\|=\sup_{i=1,2}\|T_i\|=\|T_1\|>\omega(T_1)$, therefore $\omega(T)<\|T\|$, and so T is not normaloid.

Theorem 2.11. Let T be restriction convexoid, and let $K=\{x\in H : Tx=e^{i\theta}\ T^*x$ for a fixed real value$\theta\}$. If $\sigma(T)$ is finite and $\sigma(T|K)$ lies on a convex curve. Then $\sigma_r(T|K)$ is empty.

Proof. From Theorem 2.7 $T|K$ is normal. Let μ be a complex number. Then, in view of the normality of $TK|-\mu I$, the following statements are equivalent; (1) $\mu\notin\overline{W(TK|)}$ and (2) μ is not in Conv $(\sigma(T|K))$.

Thus it follows that $\overline{W(TK)}=\mathrm{Conv}\ (\sigma(T|K))$. By assumption, $\sigma(T|K)$ lies on a convex curve and since $\overline{W(TK)}=\mathrm{Conv}(\sigma(T|K))$, we have that each point $\mu\in\sigma_p(TK|)$ must lie on the boundary of Conv $(\sigma(T|K))$ which is $\overline{W(T|K)}$, hence μ can not be an interior point of $W(T|K)$. But $\sigma_r(T|K)$ lies in the interior of $W(T|K)$, and so $\sigma_r(T|K)$ is empty.

Example 2.12. [4] Let $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and I be the one-dimensional identity operator and we consider the operator $T=A\oplus I$. Then $\overline{W(A)}=\{z : |z|\leqslant 1/2\}=\mathrm{Conv}\{\{\overline{W(A)}\},\ 1\}$, thus $\omega(T)=1=\|T\|$ and Conv $\sigma(T)=[0,\ 1]$.

III. Extreme points of $W(T)$.

Lemma 3.1. Re $T\geq 0$ if and only if $(T-\alpha I)^*(T-\alpha I)\geqslant\alpha^2$ for all $\alpha<0$.

Proof. $T\geq 0$, then for any $\alpha<0$ we have 2Re $T=T^*+T=\{T^*T-(T-\alpha I)^*(T-\alpha I)+\alpha^2 I\}/\alpha\geqslant 0\geqslant T^*T/\alpha$. Thus we have$(T-\alpha I)^*(T-\alpha I)\geq\alpha^2$ for all $\alpha<0$. Conversely, if $(T-\alpha I)^*(T-\alpha I)\geqslant\alpha^2$ for all $\alpha<0$ Then for all $\alpha<0$ we have $\alpha(T^*+T)\leqslant T^*T$, $T^*+T\geqslant\frac{1}{\alpha}\ T^*T$, and $T^*+T\geqslant 0$ results on letting $\alpha\to-\infty$; thus Re $T=1/2(T^*+T)\geq 0$, and so Re $T\geq 0$.

Lemma 3.2. Let Re $W(T)\geq 0$, and let 0 be an extreme point of $W(T)$. Then $M=\{x\in H : (Tx,\ x)=0\}$ is a closed subspace.

Proof. All is clear but the linearity. For $x,y\in M$ we see that $(T(x+y),\ (x+y))=(Tx,\ y)+(y,\ T^*x)=(Tx,\ y)+\overline{(-Tx,\ y)}=2\ \mathrm{Im}(Tx.y)\equiv a$. Assume $a\neq 0$, then $(T(e^{i\theta}x+y),\ (e^{i\theta}x+y))=2\mathrm{Im}e^{i\theta}\ (Tx,\ y)$, since $a\neq 0$, for $e=\pm 1$ the values of 2 Im $e^{i\theta}(Tx,\ y)$ lie in both the upper

and lower half-planes. Thus 0 is not an extreme point, contrary to hypothesis. Therefore $a=0$ implies $x+y\in M$. Let $\{x_n\}$ be any sequence in M such that $x_n\to x(n\to\infty)$, then we have $\lim_{n\to\infty}(Tx_n, x_n)=(T(\lim_{n\to\infty} x_n), \lim_{n\to\infty} x_n)=0$, and so $\lim_{n\to\infty} x_n=x\in M$. Thus M is closed.

Theorem 3.3 Let T be hyponormal and $(T-\alpha I)^*(T-\alpha I)\geqslant\alpha^2$ for all $\alpha<0$, where 0 is an extreme point of $W(T)$. If $(Tx, x)=0$, then $Tx=0$. Moreover, $M=\{x\in H : (Tx, x)=0\}$ is reducing subspace of T.

Proof. Let $N=\{x\in H : Tx=-T^*x\}$. Since $(Tx, x)=0$, we see that $x\in M\subset N$. From Lemma 3.1 Re $T\geqslant 0$, and so Re $W(T)\geqslant 0$. If $x\in H$, then $((T+T^*)x, x)=2((\text{Re } T)x, x)\geq 0$. Now $(Tx, x)=0$ implies $((T+T^*)x, x)=0$, and so $Tx=-T^*x$. Therefore $T|N$ is normal, by Lemma 3.2 the condition $(Tx, x)=0$ implies that $Tx=0$. Let $Tx=ay$, where $\|x\|=\|y\|=1$ and $a\neq 0$. Since $(Tx, x)=0$, it follows that $(x,y)=0$. Thus $(Ty, x)=(y, T^*x)=(y, -Tx)=-\overline{(Tx, y)}=-\bar{a}$. $\overline{(y, y)}=-\bar{a}$. Let $L=\text{span}\{x,y\}$, and define P to be the projection of H and L. Then the matrix representation of PTP on L is $\begin{pmatrix} 0 & -\bar{a} \\ a & b \end{pmatrix}$.

If $x\in N$, then $ay=Tx\in N$, hence $(Ty, y)=(y, T^*y)=(y, -Ty)=-\overline{(Ty, y)}$ and b is pure imaginary. Now, we have that $W(PTP)\subset W(T)$ and PTP is normal.

From [1] $W(PTP)$ is the line segment joining the roots of the equation $\lambda^2-b\lambda+|a|^2=0$. Now the roots are $\lambda=(b\pm i\sqrt{|b|^2+4|a|^2})/2$.

Since 0 is an extreme point of $W(T)$ and thus of $W(PTP)$, it must be an endpoint of the line segment, that is, one of the roots. Clearly this happens only if $a=0$, which implies $Tx=0$. Also, from the proof of Theorem 2.7 M is reducing subspace of T.

Corollary 3.4 [9] If T is hyponormal and z is an extreme point of $W(T)$, then $(Tx, x)=z$ together with $\|x\|=1$ implies that $Tx=zx$ and $\{x\in H : Tx=zx\}$ is a nonempty subspace.

Theorem 3.5. Let $E(T)$ be the set of all extreme points of $\overline{W(T)}$ of T on $N=\{x\in H : Tx=-T^*x\}$. If T is hyponormal, then $E(T)\cap W(T)\subseteq\sigma_p(T)$.

Proof. Let λ be given in $E(T)\cap W(T)$, then there is a unit vector z in N such that $(Tz, z)=\lambda$. By translation, we can suppose that $\lambda=0$ and Re $W(T)\geq 0$. Let $T=A+iB$ with self-adjoint, then we have $0=(Tz, z)=(Az, z)+i(Bz, z)$, and so $(Az, z)=(Bz, z)=0$. By hypothesis Re $T\geq 0$, $A=\text{Re } T\geq 0$. Hence $Az=0$. Let $M=\{x\in H : Ax=0\}$, then M is a closed subspace which contains z. By Theorem 3.3 T is normal, hence the relation $AB=BA$ holds, and so $ABx=BAx=0$ for $x\in M$, therefore $BM\subset M$ Let $C=B|M$. Then we have $(Tx, x)=(Ax, x)+i(Bx, x)=i(Cx, x)$ for a unit vector $x\in M$. Since $0\in E(T)$, either $C\geq 0$ or $B\geq 0$. From $0=(Bz, z)=(Cz, z)$ we have $Cz=Bz=0$, therefore $Tz=Az+i\ Bz=0$. Thus $0\in\sigma_p(T)$, and $E(T)\cap W(T)\subseteq\sigma_p(T)$.

Corollary 3.6. Let T be hyponormal. Then $E(T)\cap W(T)\subseteq\sigma_p(T)$.

Proof. Let $\lambda\in E(T)\cap W(T)$, we can assume that $\lambda=0$ and Re $W(T)\geq 0$. Then there is a unit vector z such that $(Tz, z)=0$. Thus $((T+T^*)z, z)=0$. Since Re $W(T)\geq 0$, this implies that $Tz=-T^*z$. Let $M=\{x\in H : Tx=-T^*x\}$, then $z\in M$, and M reduces T. In fact, if $x\in M$, then $\|Tx\|=\|T^*x\|$, hence $((T^*T-TT^*)x, x)=0$ for $x\in M$. By the the Schwars inequality, $|((T^*T-TT^*)x, y)|^2\leqslant|((T^*T-TT^*)y, y)|\cdot|((T^*T-TT^*)x, x)|=0$ for all $y\in H$, and $T^*Tx=TT^*x$ for $x\in M$, so that $T(Tx)=T(-T^*x)=-T^*(Tx)$ and $T(T^*x)=T^*(Tx)=-T^*(T^*x)$ for $x\in M$. This shows that M reduces T. Clearly the restriction $T|M$ is normal, and $0\in(W(T|M))$ is an extreme point of $\overline{W(T|M)}$ and $((T|M)z, z)=0$.

Thus $(T|M)z=Tz=0$ by Theorem 3.5.

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