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A Study on the Optimized Parametric Bispectrum Estimation

— Single-peak Bispectrum —

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<Abstract>

When analyzing various signals produced by some nonlinear processes, higher-order spectra are very useful in interpreting such signals. In this paper an optimized parametric bispectrum estimation method is presented. It is shown to be particularly advantageous when estimating from either short duration records, or under poor signal-to-noise ratio conditions. The performance is compared to a "non-optimized" parametric algorithm as well as the "direct" FFT approach when a single-peak bispectrum is estimated.

최적 파라메트릭 바이스펙트럼 추정 방법에 관한 연구

— 바이스펙트럼 피크가 1개인 경우 —

안종구 · 최재하
전자공학과

〈요 약〉

비선형 과정에 의해서 생성된 신호를 해석하고자 할 때는 고차스펙트럼을 이용하면 편리하다. 본 논문에서는 고차스펙트럼중 3차스펙트럼인 바이스펙트럼을 최적의 파라메트릭 방법으로 추정하는 방법을 보였다. 이러한 방법은 주어진 신호의 길이가 짧거나 신호대잡음비가 낮은 경우에 특히 유용하다. 발생시킨 데이타를 이용하여 본 논문에서 제안한 방법과 기존의 방법으로 바이스펙트럼을 구하여 각 방법의 장단점을 보였다.

I. Introduction

The maximum entropy method(MEM), which was introduced by Burg [1], is one of the well-known parametric methods to obtain high-resolution power spectra. In particular, the MEM has been widely used to improve the frequency resolution of the power spectrum for short time duration data records[2]. Unfortunately, it is somewhat difficult to apply the concept of the MEM to estimate higher-order spectra. The FFT method has been widely used to estimate higher-order spectra; however, it is of limited value when analyzing short duration random signals because of its relatively low frequency resolution.

Although the parametric approach to estimating higher-order spectra has not been extensively developed, Raghuveer and Nikias proposed a parametric bispectrum estimation method by using an autoregressive model[3]. Their algorithm is useful in estimating a high-resolution bispectrum for random data. It is the objective of this paper to describe an optimized parametric bispectrum estimation method and demonstrate its improved performance under relatively poor signal-to-noise ratio conditions and relatively short data lengths(say, 3dB and 1k, respectively)[4].

II. Model and Estimation

Consider a real random process $x(t)$ which is stationary to order three and zero-mean, then the triple correlation function, namely bicorrelation function, of the random process is defined by

$$R_{xxx}(\tau_1, \tau_2) = E[x(t)x(t-\tau_1)x(t-\tau_2)] \quad (1)$$

where E denotes the expected value. As shown in Eq.(1), the bicorrelation function is a function of two time delays, τ_1 and τ_2. The bicorrelation function can be defined by the 2-D inverse Fourier transform of the bispectrum, $S_{xxx}(f_1, f_2)$, in other words, these two quantities constitute a 2-D Fourier transform pair,

$$R_{xxx}(\tau_1, \tau_2) \longleftrightarrow S_{xxx}(f_1, f_2) \quad (2)$$

We start with the same model proposed by Raghuveer and Nikias[5]. Consider a real s-th order autoregressive(AR) process $x(n)$

$$x(n) + \sum_{i=1}^{s} a_i x(n-i) = e(n) \quad (3)$$

where the $e(n)$'s are i.i.d. and zero-mean non-Gaussian(i.e., $E[e^3(n)] = b \neq 0$) and $x(m)$ is independent of $e(n)$ for $m<n$. We assume that $e(n)$ is third-order stationary. Multiplying both sides of Eq.(3) by $x(n-j)x(n-k)$ and then taking the expectation, we obtain

$$R(j, k) + \sum_{i=1}^{s} a_i R(j-i, k-i) = \beta\delta(j, k) \quad (4)$$

where $R(m, n)$ is the third-order correlation function of the process, $\delta(j, k)$ is the 2-D unit impulse function and j=0, 1, 2, ..., s and k=0, 1, 2, ..., s; however, $j \leq k$ due to the symmetry property of the bispectrum. Although the number of unknowns is s+1 including b the number of independent AR bicorrelation equations is $\frac{(s+1)(s+2)}{2}$. Since the number of equations is greater than the number of

unknowns, it is overdetermined. We may thus rewrite Eq.(4) in matrix form as follows:

$$\mathbf{ra} = \mathbf{b} \tag{5}$$

where

$$\mathbf{r} = \begin{bmatrix} R(0,0) & R(-1,-1) & \dots & R(-s,-s) \\ R(0,1) & R(-1,0) & \dots & R(-s,1-s) \\ \dots & & \dots & \\ R(0,s) & R(-1,s-1) & \dots & R(-s,0) \\ R(1,1) & R(0,0) & \dots & R(1-s,1-s) \\ R(1,2) & R(0,1) & \dots & R(1-s,2-s) \\ \dots & & \dots & \\ R(s,s) & R(s-1,s-1) & \dots & R(0,0) \end{bmatrix}, \tag{6}$$

$$\mathbf{a} = [\,1\ a_1\ a_2 \dots a_s\,]^T \tag{7}$$

and

$$\mathbf{b} = [\beta\ 0\ 0 \dots 0\,]^T \tag{8}$$

The dimension of the matrix $\mathbf{r}$ is $\frac{(s+1)(s+2)}{2} \times (s+1)$, that of $\mathbf{a}$ is $(s+1)\times 1$ and that of $\mathbf{b}$ is $\frac{(s+1)(s+2)}{2}\times 1$. Since there are $s+1$ unknowns, only $s+1$ equations which are on the $j=k$ line are chosen among $\frac{(s+1)(s+2)}{2}$ equations in order to calculate the unknowns(i.e., parameters) in the former parametric method [5]. Generally speaking, however, it is not easy to state that the parameters which are obtained by using the former method also satisfy the other equations which are on the $j \neq k$ lines. Therefore, we utilize a multiple regression method to provide a best-fitting plane to the data, that is, we obtain the least squares estimates of the a_i's, which are optimum in the least square sense. Thus, by solving the following matrix equation, one can obtain the least squares estimates of the a_i parameters [6],

$$\mathbf{r}^T\mathbf{ra} = \mathbf{r}^T\mathbf{b}. \tag{9}$$

Finally, the bispectrum is given by

$$S_{xxx}(f_1, f_2) = \beta H(f_1)H(f_2)H^*(f_1+f_2) \tag{10}$$

where the AR filter transfer function $H(f)$ is The details surrounding Eqs.(10) and (11) are

$$H(f) = [\,1 + \sum_{i=1}^{s} a_i exp(-j2\pi fi)]^{-1} \tag{11}$$

described in reference [5].

Although all elements except the first element of $\mathbf{b}$ of the model are zeros, those of the estimated $\hat{\mathbf{b}}$ may not be zeros. In order to show the goodness of fit of the plane, on can use the variance of $\hat{\mathbf{b}}$, which is given by[7]

$$VAR_b = \frac{\sum (\hat{b}_i - b_i)^2}{\frac{(s+1)(s+2)}{2} - 1 - s} \tag{12}$$

where $\hat{b}_i$ and b_i are the i-th elements of $\hat{\mathbf{b}}$ and $\mathbf{b}$, respectively.

III. Experimental Results

The bispectrum is obtained by using the following three techniques in this section: (1) the optimized AR method(OARM) which is presented in this paper, (2) the non-optimized AR method(NOARM) proposed by Raghuveer and Nikias[5], and (3) the FFT

method [8]. The experiments are carried out for data lengths of 8k, 2k and 1k and signal-to-noise ratios (SNR's) of 30, 10 and 3dB. Let the number of realizations be M and the number of data points in each realization be N. When we use the AR methods, the maximum model order is N-1. For example, the maximum model order is 63, when N=64. Since there is not a method to determine the optimum model order in order to parametically estimate the bispectrum, the experiments are carried out for all model orders ranging from 3 to N-1. In this section we compare the ability of the three bispectral estimators to locate a single peak in the two-dimensional bispectral frequency plane.

Consider the signal given by

$$x(t) = \sum_{i=1}^{6} \cos(2\pi g_i t + \theta_i) + n(t) \qquad (13)$$

where $g_1 = 0.34Hz$, $g_2 = 0.15Hz$, $g_3 = 0.49Hz$, $g_4 = 0.37Hz$, $g_5 = 0.06Hz$ and $g_6 = 0.43Hz$, and $\theta_1+\theta_2 = \theta_3$ and $\theta_4+\theta_5 \neq \theta_6$, that is, the former three sinusoids are correlated (because of their phase coherence), but the latter three sinusoids are uncorrelated(no phase coherence). The Nyquist frequency is $0.64Hz$, the number of realizations(M) is 64 and the number of data points in each realization(N) is 128, i.e., the total number of data points is 8k.

For the purposes of this experiment, let us define the 'power' (square of the absolute value of the bispectrum) ratio of bispectrum (PRB) on the bispectrum plane in order to facilitate comparison of the performance of the three methods. Since, for this example, the ture bispectrum is zero everywhere except at(0.34Hz, 0.15Hz) on the bispectrum plane, all non-zero bispectral estimates except the true bispectrum may be regarded as noise. The PRB on the bispectrum plane is defined by the ratio of the "power" of the true bispectrum to the sum of the "powers" of all spurious bispectra on the bispectrum plane except at the true bispectrum.

A method to determine the appropriate order, that is, s in Eq.(4), of the system is not well established. The PRB on the bispectrum plane is used as a criterion to determine the order of the system in these experiments. In particular, the PRB's on the bispectrum plane are calculated for systems for which the orders vary from 3 to N-1. Then the order of the system, for which the PRB on the bispectrum plane is largest, is selected.

When the signal of Eq.(13) has a SNR of 30dB, all methods give comparable results. When the SNR is reduced to 3dB, the PRB on the bispectrum plane obtained using the OARM with model order 26 is 55.3dB, the PRB using the NOARM with model order 42 is 49.2dB and the PRB using the FFT is 42. 8dB. Plots of the three bispectra are shown in Fig.1. Thus the performance of the OARM (Fig.1(a)) is somewhat better than those of the other two (Fig.1(b) and (c)), although all three yield "acceptable" results in this case. The OARM can locate the peak for all model orders from 7 to 63, however, the NOARM can find the peak for some model orders, for example, 21, 22, 25, 28 to 31, 35 to 56 and 60 to 63. For other orders, we typically observe either the true peak is very small

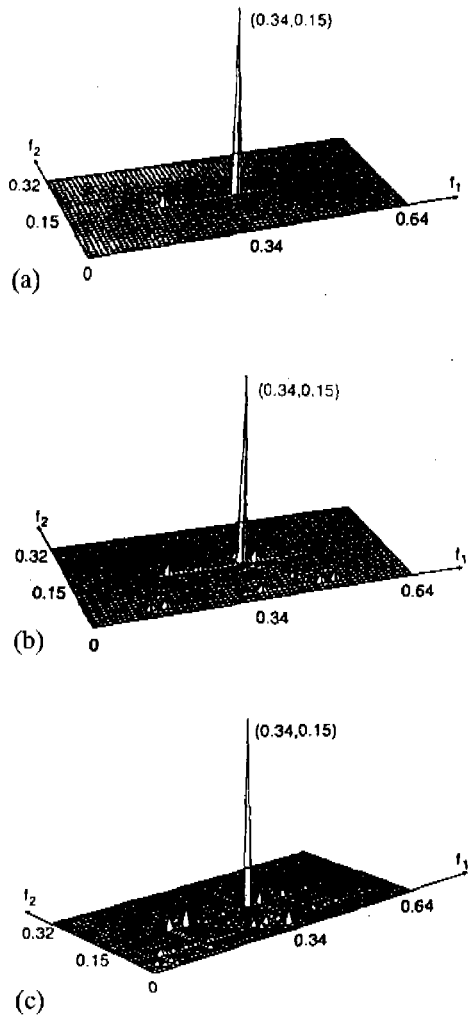

Fig.1 Bispectrum(SNR=3dB, data length = 8k, # realization =64, # data points per realization=128)
(a) OARM(model order = 26)
(b) NOARM(model order = 42)
(c) FFT method

with many spurious peaks present, or a large single peak dominates but at the wrong frequency coordinates. Note that N=128 in this case, thus the maximum model order is 127(i.e., N-1), however, the experiments are carried out only up to model order of 63, since the data length is sufficiently long that both the OARM and the NOARM could find the true bispectra at model orders less than 63.

Next, consider the case where the SNR of the signal in Eq.(13) is 3dB and the total number of data points is decreased from 8k to 1k, i.e., the number of realizations(M) is 16 and the number of data points in each realization(N) is 64. The three bispectra of the noisy short duration data record are shown is Fig.2. If we use the OARM with model order 15 to estimate the bispectrum, we find only one peak, shown in Fig.2(a), at the correct coordinates in the bispectrum plane with a PRB = 36.7dB in this case. Under these same conditions, the NOARM with model order 50 can find the exact frequency coordinates of the true bispectrum with a PRB=33.3dB, however, spurious peaks also appear as indicated in Fig.2(b). The FFT approach with a PRB=20.9dB also yields several spurious peaks as indicated in Fig.2(c). Because of resolution limitations, the FFT approach can not find the exact peak in this case since N is 64. The frequency coordinates of the largest peak among several peaks on the bispectrum plane with the FFT method is not(0.3*Hz*, 0.15*Hz*), but (0.34*Hz*, 0.14*Hz*) as shown in Fig.2(c). The OARM can correctly locate the peak for all model orders from 14 to 53, however, the NOARM can correctly locate the peak only for model orders 47, 48, 50, 53 to 55 and 57 in this example.

Since the bispectra of the OARM and

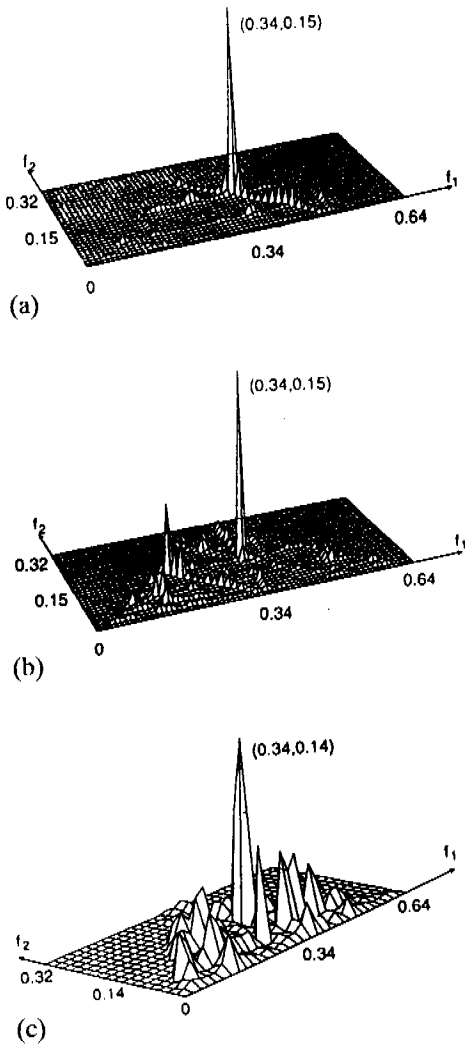

Fig.2 Bispectrum(SNR=3dB, data length = 1k, # realization =16, # data points per realization=64)
(a) OARM(model order = 15)
(b) NOARM(model order = 50)
(c) FFT method

NOARM are obtained from the AR filter transfer function $H(f)$ in Eq.(11), the AR filter transfer functions of both the methods are obvious of great importance. Amplitude

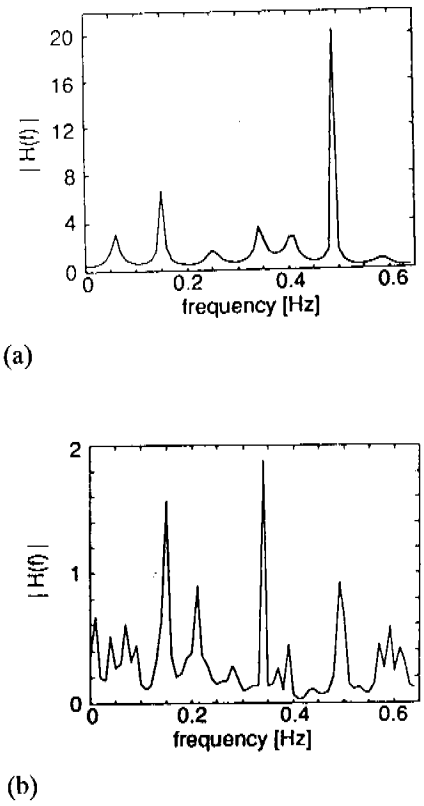

Fig.3 Amplitudes of the AR filter transfer functions(SNR=3dB, data length = 1k, # realization =16, # data points per realization=64)
(a) OARM(model order = 15)
(b) NOARM(model order = 50)

of the AR filter functions of the OARM and NOARM methods for the 1k example above are shown in Fig.3(a) and (b), respectively. As shown in Fig.3(b), there are many spurious peaks which yield many spurious peaks in the bispectrum plane. One of the reasons why the AR filter transfer function of the NOARM has more spurious peaks than that of the OARM is the fact that, generally speaking, the NOARM requires a higher

model order than the OARM in order to estimate a "good" bispectrum under the same conditions.

As a result of many experiments that we have carried out, we observe that, if the model order is too high, the OARM and NOARM could not find the true peaks as stated previously in this 1k example, since autobicorrelations, R(j, k)'s, with higher j and k are required. However, these autobicorrelations are less valuable in the segment averaging method, since, for example, when N=64, R(63, 63) is calculated without averaging within each realization. (Recall that a model order of 63 is the largest model order one can take when N=64). To summarize, the error which is involved in calculating R(j, k) with higher j and k inhibits one from obtaining good bispectrum estimates with higher-order models.

Next, let us consider the goodness of fit of the model by using the variance of $\hat{\mathbf{b}}$, given by Eq.(12), which is the variance of the estimation errors. As indicated in Fig.4, the variance of the estimation errors of the NOARM is much larger than the estimation errors of the OARM. Presumably this due to the coefficients using the NOARM satisfy only the bicorrelation equations on the $j=k$ line, not the equations on the $j\neq k$ lines. This is one reason why it is desirable to use multiple regression analysis in order to estimate the bispectrum.

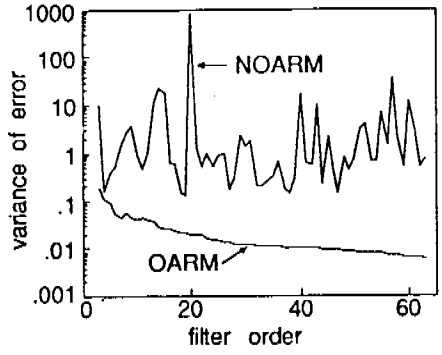

Fig.4 Variances of estimation errors vs. filter order (SNR=3dB, data length=1k, # realization=16, #data points per realization=64)

IV. Conclusion

Obviously, the results represented in Section III are not general since they are based on specific simulation experiments. However, they are useful in terms of providing some insight into the advantages and limitations of the optimized parametric estimator discussed in this paper. On the basis of this initial investigation, it appears that the OARM is a particularly useful approach for estimating bispectra when dealing with relatively low SNR and short duration data records. On the other hand, when the data records are sufficiently long and the SNR sufficiently high, the OARM, the NOARM and the FFT method yield comparable results. In the intervening region of moderately long data records and moderate SNR's, the OARM and NOARM yield comparable and acceptable results, whereas the FFT method tends to yield

unacceptable results.

One of the problems of the parametric bispectrum estimation methods, OARM and NOARM, is concerned with determining the "best" model order. This is a particularly challenging issue when dealing with field test data since the "true" bispectrum is not known beforehand. Fortunately, OARM bispectral estimation appears to be less sensitive to the model order.

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