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지연제어동작을 갖는 샘플데이타 시스템의 최적제어

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<요 약>

본 논문에서는 지연제어동작을 갖는 샘플데이타 시스템의 제어문제를 다루고자 한다. 그 문제는 전적으로 이산시간 영역에서 명확히 설명되고 선형최적확률제어기법(LQG approach)에 의하여 해결된다. 또한 본 논문에서 제안된 제어기법(LQG approach)으로 설계된 선형최적확률보상기(LQG compensator)가 지연제어입력을 가지는 경우에도 분리원리(Separation Principle)가 성립됨을 증명한다. 본 연구의 이론적인 결과를 Two-Mass-Spring 시스템에 적용하여 검증하였다. 결과로부터 알 수 있는 사실은 지연제어동작으로 야기된 시스템의 안정성(stability)과 강건성(robustness) 그리고 제어시스템의 성능(performance) 저하는 본 논문에서 제안된 방법을 통하여 회복 또는 향상될 수 있다는 것이다. 특히 그 최적확률보상기는 시간지연으로 발생하는 제어동작 포화(saturation)로 부터 샘플데이타 시스템을 보호할 수 있다.

Optimal Control of Sampled-Data System with Delayed Control Action

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<Abstract>

In this paper, we treat the sampled-data control problem that accounts for time-delay in the control action. The problem is formulated entirely in discrete-time domain, and the optimal design is achieved in Linear Quadratic Gaussian(LQG) approach. It is shown in this approach that 'Separation Principle' is valid with an additional feedback of delayed control input to the LQG compensator proposed in this paper. The method is applied to a benchmark problem of two-mass-spring system. Design results obtained indicate that degradation of stability, robustness and performance due to the delayed control action may get recovered to (or even better than) level of the system with no delay time through the design method proposed. Especially, the LQG approach shows the fact that the LQG compensation may prevent the sampled-data control system from saturation of control action due to the time-delay.

I. Introduction

With development of digital computer, computer-based control systems have gain broad acceptance to industrial applications. Digital controllers have many advantages over their analog counterparts; easy reprogramming without expensive wiring changes, smaller size and light weight, and cheaper than analog devices. However, digital controllers pose some fundamentally different characteristics that should be carefully examined. One of them is, what we call, computation time-delay. This is the main issue to be addressed in this paper. Problems related to the time-delay have been extensively studied. Dorato and Levis [1] showed the relevant transformation of reducing a sampled-data control problem into an equivalent discrete-time control problem. Mukhopadhyay [2] proposed a optimal design method for a digital control system using constrained optimization. However, the method did not address the problem of computation time-delay. In [3], the authors examined the time-delay for an optimal state-feedback problem. Their consideration is not practical because full-state information may not always be available so that the feedback controller designed can not be implementable. Mita [4] formulated and solved an optimal problem with computation time-delay which is an integer multiple of the sampling time. Diduch and Doraiswami [5] pursued the problem of control servomechanism design. Ha and Ly [6],[7] proposed a new approach of digital control synthesis using parameter optimization and solved the problem of computation time-delay in a suboptimal classical controller design [8]. In this paper, we deal with a sampled-data system with the time-delay at control action which is *less than a sampling time*. For more practicality, the optimal control problem with measurement-based output-feedback is considered. The problem formulation is based on linear quadratic gaussian (LQG) framework for digital control-law design.

This paper is organized as follows. In Section II, the control problem taking into account the time-delay at control action is defined. Section III presents a state-space formulation of the problem in the discrete-time domain and its solution of the problem in LQG framework. In Section IV, the problem with measurement-based output-feedback is solved. Results applied to the benchmark problem of a two-mass-spring system are discussed in Section V, and conclusions are summarized in Section VI.

II. Control Problem with the Time-Delay

Consider a linear time-invariant system with zero initial condition,

$$\Sigma_{sz}: \begin{cases} \dot{x}(t) = A\,x(t) + B_1\,w_c(t) + B_2\,u(t) \\ z(t) = C_1\,x(t) + D_{12}\,u(t) \\ yk = H_d\,x_k + D_2\,v_k \end{cases} \quad (1)$$

where $x(t)$ is a state vector in $\Re^n$, $u(t)$ a control input vector in $\Re^m$, reconstructed by digital-to-analog converter (DAC), and $z(t)$ a controlled output vector in $\Re^q$. Usually y_k, x_k, u_k, and v_k denote the responses of $y(t)$, $x(t)$, $u(t)$ and $v(t)$ at each sampling time, respectively. The output y_k is a discrete-time measurement vector in $\Re^p$, resulted from analog-to-digital converter (ADC). And $w_c(t)$ is a continuous-time process disturbance vector in $\Re^{d_1}$ and v_k is a discrete-time measurement noise vector in $\Re^{d_2}$. The matrices A, B_1, B_2, C_1, D_{12}, H_d and D_2 are constant matrices with appropriate dimensions. We assume that the pair $\{A, B_2\}$ is stabilizable, $\{A, B_1\}$ is disturbable, the pairs $\{A, C_1\}$ and $\{A, H_d\}$ are both detectable. Also it is assumed for causality that $u(t)=0$ for $t \leq 0^-$. The sampling time T is assumed properly given.

The time-delay at control action less than a sampling time T is taken into account in the control synthesis. So the time-delay is explicitly defined at the feedback control action.

$$u(t) = \begin{cases} u_{k-1} & kT \leq t < kT + \lambda \\ u_k & kT + \lambda \leq t < (k+1)T \end{cases} \quad (2)$$

where λ is an *a-priori known* time-delay less than T. A block-diagram of the sampled-data system with the time-delay at control action is depicted in Figure 1. Let's consider the following quadratic cost function J_c for the system Σ_{sz},

$$J_c = \lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} E\,\{z^T(t)Qz(t) + u^T(t)Ru(t)\}\,dt \quad (3)$$

where $E\{\cdot\}$ denotes the expectation operator and $z^T(t)$ is transpose of the vector $z(t)$. The matrices Q and R in the cost function J_c are symmetric positive definite weighting matrices. Our problem to be concerned is associated with a measurement-based feedback control in which an output-feedback controller is designed as follows

$$\begin{cases} x_{ck+1} = A_c x_{ck} + B_c yk \\ u_k = C_c x_{ck} + D_c yk \end{cases} \quad (4)$$

where the controller is linear shift-invariant. Note that although the generic nature of sampled-data system is time-varying due to periodic sample and hold operation, our concern is focused on a linear time-invariant controller design.

Our problem to be solved in this paper is to find the optimal digital controller as shown in Eq.(4) that minimizes the quadratic cost function J_c for a sampled-data system with the computation time-delay defined in Eq.(2)

under stochastic environment.

III. Formulation in Linear Quadratic Gaussian Framework

Suppose that the disturbance $w_c(t)$ and the measurement noise v_k coming into the sampled-data system are the random process with the following stochastic properties.

$$E\{w_c(t)\} = \underline{0}\,,\ E\{w_c(t)w_c^T(\tau)\} = W_c\delta(t-\tau)$$
$$E\{v_k\} = \underline{0}\,,\quad E\{v_k v_j^T\} = V_o\delta_{kj} \tag{5}$$

where W_c and V_o denote covariances of the continuous-time process disturbance $w_c(t)$ and the discrete-time measurement noise v_k, respectively. Note that $\delta(t-\tau)$ and δ_{kj} denote the well-known Kroneckar delta function; for instance, $\delta(t-\tau)$=1 (or δ_{kj}=1) for $t=\tau$ (or $k=j$) otherwise zero.

For a compact description of the sampled-data system in Eq.(1) at a sampling time kT, let us introduce the following definitions.

Definition III.1

$$\Phi(t) = e^{At}\,,\quad \Psi(t) = \int_0^t e^{A\theta}\,d\theta B_2$$
$$G_d\xi_k = \int_0^T \Phi(T-\theta)B_1 w_c(kT+\theta)\,d\theta \tag{6}$$

where $w_c(t)\in L_2[0,T]$, and ξ_k is a discrete sequence vector in $\Re^{n\times 1}$ that belongs to a class of l_2 or of gaussian white sequences with zero mean and identity covariance. And G_d is a linear transformation defined as follows.

$$G_d : L_2[0,T] \rightarrow \Re^n$$

Namely, we treat the class of L_2-disturbances $w_c(t)$ which are square-integrable in the time interval $t\in[0,T]$ or of gaussian white randon processes.

Now, let us describe the sampled-data system with the delayed control action at a sampling time kT. According to Ref.[1], the sampled-data system in Eq.(1) can be properly discretized with the time-delay as defined in Eq.(2). For simplicity, by introducing a hold state x_k^h which is equal to u_{k-1}, let us define an augmented state x_k^λ .

$$x_k^\lambda := \begin{bmatrix} x_k \\ x_k^h \end{bmatrix}$$

Then we can represent the desired description of the sampled-data system in state-space.

$$\Sigma_z^\lambda : \begin{cases} x_{k+1}^\lambda = A^\lambda x_k^\lambda + B_1^\lambda \xi_K + B_2^\lambda u_k \\ zk = C_1^\lambda x_k^\lambda + D_{12}u_k \\ yk = H_d^\lambda x_k^\lambda + D_2 v_k \end{cases} \tag{7}$$

where

$$A^\lambda = \begin{bmatrix} \Phi(T) & \Psi(T)-\Psi(T-\lambda) \\ \underline{0}_{m\times n} & \underline{0}_{m\times m} \end{bmatrix}$$
$$B_1^\lambda = \begin{bmatrix} G_d \\ \underline{0}_{m\times d_1} \end{bmatrix},\quad B_2^\lambda = \begin{bmatrix} \Psi(T-\lambda) \\ I_{m\times m} \end{bmatrix} \tag{8}$$
$$C_1^\lambda = [C_1\ \ \underline{0}_{q\times m}]\,,\qquad H_d^\lambda = [H_d\ \ \underline{0}_{p\times m}]$$

Note that it is assumed that $E\{\xi_k v_k^T\}=\underline{0}$.

Also the cost function J_c in Eq.(3) is properly discretized with intersample behaviour resulted from effect of the time-delay. For simplicity, define a new function.

$$\overline{\Psi}(t) := \int_0^t e^{-A\theta}\, d\theta B_2$$

Then the resulted cost function $J(u_k, \lambda)$ is represented as follows.

$$J(u_k, \lambda) = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=0}^{N} E\left\{ \begin{bmatrix} x_k^\lambda \\ u_k \end{bmatrix}^T \begin{bmatrix} \overline{Q} & \overline{M} \\ \overline{M}^T & \overline{R} \end{bmatrix} \begin{bmatrix} x_k^\lambda \\ u_k \end{bmatrix} \right\} \tag{9}$$

where

$$\overline{Q} := \begin{bmatrix} Q_1 & N_1 \\ N_1^T & R_1 \end{bmatrix}, \quad \overline{M} := \begin{bmatrix} N_2 \\ R_2 \end{bmatrix} \tag{10}$$

For complete expression of the cost function $J(u_k, \lambda)$, let us introduce the following definitions.

$$Q_{xx}(a, b) := \frac{1}{T}\int_a^b \Phi(\theta)^T C_1^T Q C_1 \Phi(\theta)\, d\theta$$

$$Q_{xu}(a, b) := \frac{1}{T}\int_a^b \Phi(\theta)^T C_1^T Q C_1 \Psi(\theta)\, d\theta$$

$$Q_{uu}(a, b) := \frac{1}{T}\int_a^b \Psi(\theta)^T C_1^T Q C_1 \Psi(\theta)\, d\theta$$

$$N_{xu}(a, b) := \frac{1}{T}\int_a^b \Phi(\theta)^T C_1^T Q D_{12}\, d\theta$$

$$N_{uu}(a, b) := \frac{1}{T}\int_a^b \Psi(\theta)^T C_1^T Q D_{12}\, d\theta$$

Then the weighting matices in Eq.(9) are expressed in detail.

$$Q_1 = Q_{ss}(0,T)$$

$$N_1 = Q_{xu}(0,\lambda) + Q_{xx}(\lambda,T)\overline{\Psi}(\lambda) + N_{xu}(0,\lambda)$$

$$R_1 = Q_{uu}(0,\lambda) + \overline{\Psi}(\lambda)^T Q_{xx}(\lambda,T)\overline{\Psi}(\lambda) + N_{uu}(0,\lambda) + N_{uu}(0,\lambda)^T + (R + D^T QD)\frac{\lambda}{T}$$

$$N_2 = \Phi(\lambda)^T \{Q_{xu}(0,T-\lambda) + N_{xu}(0,T-\lambda)\}$$

$$R_2 = \overline{\Psi}(\lambda)^T \Phi(\lambda)^T \{Q_{xu}(0,T-\lambda) + N_{xu}(0,T-\lambda)\}$$

$$\overline{R} = Q_{uu}(0,T-\lambda) + N_{uu}(0,T-\lambda) + N_{uu}(0,T-\lambda)^T + (R + D_{12}^T Q D_{12})(1 - \frac{\lambda}{T})$$

Note that the cost function $J(u_k, \lambda)$ includes sensitivity of intersample behavior due to the time-delay.

Nest, with the formulation shown in this section, we would like to solve the problem of sampled-data system with the delayed control action in the measurement-based feedback compensation within LQG framework.

IV. A Measurement-Based Output-Feedback Compensation

In this section, we would like to solve the problem addressed in linear quadratic gaussian design approach. First of all, consider conditions of controllability (or stabilizability) and observability (or detectability). From Ref. [3], we know that the sampled-data system in Eq.(7) is controllable (observable) if and only if the system in Eq.(1) is controllable (observable) under proper selection of sampling time T. Next, assume that the well-known 'separation principle' is valid; however it will be proved to be valid later. To design the *optimal* linear quadratic gaussian compensator under this assumption, the optimal full-state feedback law (resulted from discrete-time Euler-Lagrange Equation; Ref.[9]) and the optimal estimator (resulted from sequential dynamic programming; Ref.[10]) are *separately* obtained. So let us discuss about how to obtain the optimal full-state feedback control law. From the well-known linear quadratic regulator theory, the optimal full-state feedback law for the time-delay problem being considered can be expressed by the following.

$$u_k = K^{\lambda} x_k^{\lambda} := K^{x} x_k + K^{h} u_{k-1} \tag{11}$$

There are two methods to obtain the full-state feedback law K^{λ}. One is to use the eigenvector decomposition of discrete-time Hamiltonian matrix and the other the iterative method of the discrete-time Riccati equation. However, the eigenvector decomposition method is *not* applicable to this problem because the system matrix A^{λ} in Eq.(7) is singular. Therefore, this problem being considered can only be solved through the iterative method. The recursive discrete-time Riccati equation for this time-delay problem is resulted from the discrete-time Euler-Lagrange Equation.

$$S_k = A_c^{\lambda^T} \Lambda_{k+1} A_c^{\lambda} + \tilde{Q} \tag{12}$$

where

$$\Lambda_{k+1} := S_{k+1} - S_{k+1} B_2^{\lambda} \overline{R}^{-1} B_2^{\lambda^T} S_{k+1}$$

$$A_c^{\lambda} := A^{\lambda} - B_2^{\lambda} \overline{R}^{-1} \overline{M}^{T}$$

$$\tilde{Q} := \overline{Q} - \overline{M}\,\overline{R}^{-1} \overline{M}^{T}$$

Note that this Riccati equation is time-varying so that in order to obtain the steady-state solution we have to solve it iteratively *backward in time*. The resulted steady-state solution becomes

$$S^{\lambda} = \lim_{k \to -\infty} S_k \tag{13}$$

Hence the steady-state optimal full-state feedback control law becomes

$$K^{\lambda} = -\overline{R}^{-1} \{ B_2^{\lambda} A_c^{\lambda^{-T}} (S^{\lambda} - \tilde{Q}) - \overline{M}^{T} \} \tag{14}$$

For complete design of the optimal compensator with measurement-based feedback to be achieved, let us concern about design of the optimal estimator which reconstructs unavailable states information from the discrete measured output sequences y_i for i=1,2,...,k. Note that D_2 has a full-row rank. Let us define the best estimated state as $\hat{x}_k := E\{x_k | y_i, u_i \ \forall i=1,2,\ldots,k\}$ where E{ · | · } means conditional expectation operator. The following optimal estimator is proposed for the time-delay problem being concerned.

$$\left\{ \begin{aligned} \overline{x}_{k+1} &= \Phi(T)\hat{x}_k + \Psi(T-\lambda)u_k + \{\Psi(T) - \Psi(T-\lambda)\}u_{k-1} \\ \hat{x}_k &= \overline{x}_k + L_c \{ y_k - H_d \overline{x}_k \} \end{aligned} \right\} \tag{15}$$

where $\overline{x}_k := E\{x_k | y_i, u_i \ \forall i=1,2,\ldots,k-1\}$. The optimal estimator gain matrix L_c is obtained via discrete-time sequencial dynamic programming.

$$L_c = M H_d^T (H_d M H_d^T + D_2 V_o D_2^T)^{-1} \qquad (16)$$

where the steady-state error covariance M is defined as

$$M := \lim_{k \to \infty} E\{(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T\}$$

while satisfying the following Riccati equation

$$\begin{aligned} M &= \Phi(T) P \Phi(T)^T + G_d G_d^T \\ P &= M - M H_d^T (H_d M H_d^T + D_2 V_o D_2^T)^{-1} H_d M \end{aligned} \qquad (17)$$

where the steady-state error covariance P is defined as

$$P := \lim_{k \to \infty} E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\}$$

Note that the discrete-time Riccati equation in Eq.(17) can be solved in the eigenvector decomposition of discrete-time Hamiltonian matrix. In order to implement the optimal full-state feedback law in Eq.(11) through measurement-based output feedback compensation, the estimated state $\hat{x}_k$ substitutes the state x_k in Eq.(11).

$$u_k = K^\lambda \hat{x}_k^\lambda := K^x \hat{x}_k + K^h u_{k-1} \qquad (18)$$

With Eq.(18), the following optimal compensator is obtained.

$$K_z^\lambda : \left\{ \begin{aligned} \bar{x}_{k+1} &= A_z^\lambda \bar{x}_k + B_z^\lambda \begin{bmatrix} y_k \\ u_{k-1} \end{bmatrix} \\ u_k &= C_z^\lambda \bar{x}_k + D_z^\lambda \begin{bmatrix} y_k \\ u_{k-1} \end{bmatrix} \end{aligned} \right\} \qquad (19)$$

where

$$\begin{aligned} A_z^\lambda &:= \{\Phi(T) + \Psi(T-\lambda) K^x\}\{I_n - L_c H_d\} \\ B_z^\lambda &:= [\{\Phi(T) + \Psi(T-\lambda) K^x\} L_c \quad \Psi(T) + \Psi(T-\lambda)\{K^h - I_m\}] \\ C_z^\lambda &:= K^x \{I_n - L_c H_d\} \\ D_z^\lambda &:= [K^x L_c \quad K^h] \end{aligned}$$

From the optimal compensator, there is an interesting observation.

Remark IV.1 *The optimal compensator proposed in Eq.(19) is different in its structure from conventional discrete-time optimal compensator with no time-delay ($\lambda = 0$). The proposed compensator has an additional input term of the delayed control input u_{k-1}. The role of this term may be to compensate the delayed effect on stability, performance and robustness of the sampled-data system with the delayed control action.*

Once we obtain the optimal compensator in Eq.(19), we evaluate the closed-loop system. For simplicity, let us define the following vectors

$$x_k^{cl} := \begin{bmatrix} x_k^\lambda \\ \bar{x}_k \end{bmatrix}, \quad \eta_k := \begin{bmatrix} \xi_k \\ v_k \end{bmatrix}$$

For analysis of the sampled-data system, the closed-loop system is obtained by feeding back the compensator to the discrete-time system in Eq.(7) and depicted in Figure 2.

$$x_{k+1}^{cl} = A_c^{cl} x_k^{cl} + G_c^{cl} \eta_k \tag{20}$$

where

$$A_c^{cl} := \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}, \quad G_c^{cl} := \begin{bmatrix} G_{c11} & G_{c12} \\ G_{c21} & G_{c22} \end{bmatrix}$$

with

$$\begin{aligned}
A_{c11} &:= [\Phi(T) + \Psi(T-\lambda)K^x L_c H_d \; K^x L_c H_d \\
&\qquad \Psi(T-\lambda)\{K^h - I_m\} + \Psi(T)K^h] \\
A_{c12} &:= [\Psi(T-\lambda)K^x \{I_n - L_c H_d\} \\
&\qquad K^x \{I_n - L_c H_d\}] \\
A_{c21} &:= [\{\Phi(T) + \Psi(T-\lambda)K^x\} L_c H_d \\
&\qquad \Psi(T-\lambda)\{K^h - I_m\} + \Psi(T)] \\
A_{c22} &:= [\{\Phi(T) + \Psi(T-\lambda)K^x\} \\
&\qquad \{I_n - L_c H_d\}] \\
G_{c11} &:= \begin{bmatrix} G_d \\ \underline{0}_{m\times n} \end{bmatrix}, \quad G_{c11} := \begin{bmatrix} \Psi(T-\lambda)K^x L_c D_2 \\ K^x L_c D_2 \end{bmatrix} \\
G_{c21} &:= [\underline{0}_{n\times n}], \\
G_{c22} &:= [\{\Phi(T) + \Psi(T-\lambda)K^x\} L_c D_2]
\end{aligned}$$

With the closed-loop system analysis, 'Separation Principle' for the time-delay problem being addressed is proved in the following theorem, which is an important basis for design of the optimal compensator in Eq.(19).

Theorem IV.1 *With the compensator proposed in Eq.(19), the 'separation principle' is still valid for the sampled-data system with the delayed control action given in Eq.(2).*

Proof: Let us introduce a nonsingular transformation defined as

$$\begin{bmatrix} x_k^\lambda \\ \tilde{x}_k \end{bmatrix} := \begin{bmatrix} I_{n+m} & \underline{0}_{(n+m)\times n} \\ [I_{n\times n} \;\; \underline{0}_{n\times m}] & -I_{n\times n} \end{bmatrix} \begin{bmatrix} x_k^\lambda \\ \tilde{x}_k \end{bmatrix}$$

After applying the transformation to the closed-loop system in Eq.(20), we obtain the closed-loop system matrix A_c^{cl} equivalent to the following.

$$A_c^{cl} \sim \begin{bmatrix} A^\lambda + B_2^\lambda K^\lambda & -B_2^\lambda K^x \{I_n - L_c H_d\} \\ \underline{0}_{n\times(n+m)} & \Phi(T)\{I_n - L_c H_d\} \end{bmatrix}$$

It is obvious that when the compensator proposed in Eq.(19) is applied to the sampled-data system the separation principle still holds under existence of the computation time delay given in Eq.(2).

Next, in order to analyze the sampled-data system with the computation time-delay, the steady-state covariance X^{cl} of the closed-loop system in Eq.(20) is obtained.

$$X^{cl}, = A_c^{cl} X^{cl} A_c^{cl^T} + G_c^{cl} \begin{bmatrix} I_n & \underline{0} \\ \underline{0} & V_o \end{bmatrix} G_c^{cl^T} \tag{21}$$

where the state covariance X^{cl}, in a sense of time average, is defined as

$$X^{cl} := \lim_{N\to\infty} \frac{1}{2N} \sum_{k=0}^{N} E\{x_k^{cl} x_k^{cl^T}\}$$

With the state covariance X^{cl} given in Eq.(21), we obtain the optimum cost $J^o(\lambda)$ of the cost function in Eq.(9) for this LQG compensation.

$$\hat{J}(\lambda) = Trace\left\{\begin{bmatrix} (\bar{Q}+\bar{M}K^{\lambda})X^{\lambda} & \bar{Q}X^{\lambda}K^{\lambda^T} + \\ & \bar{M}(K^{\lambda}X^{\lambda}K^{\lambda^T} + K^{x}PK^{x^T}) \\ (\bar{M}^T+\bar{R}K^{\lambda})X^{\lambda} & \bar{M}^T X^{\lambda}K^{\lambda^T} + \\ & \bar{R}(K^{\lambda}X^{\lambda}K^{\lambda^T} + K^{x}PK^{x^T}) \end{bmatrix}\right\} \tag{22}$$

where $Trace\{\cdot\}$ implies the matrix operation of trace, X^{λ} denotes a part of the state covariance X^{el} given in Eq.(21), which is defined as

$$X^{\lambda} := \lim_{N\to\infty} \frac{1}{2N}\sum_{k=0}^{N} E\left\{\begin{bmatrix} x_k \\ x_k^h \end{bmatrix}\begin{bmatrix} x_k \\ x_k^h \end{bmatrix}^T\right\}$$

and the error covariance P is given in Eq.(17).

In next section, we would like to apply the LQG design approach developed in the previous sections to a benchmark problem of two-mass-spring system.

V. A Benchmark Problem: Two-Mass-Spring System

So far we considered a fundamental problem occurred in sampled-data systems which takes into account the delayed control action due to computation time-delay. The problem is formulated in state space and entirely in the discrete-time domain. As a solution method the linear quadratic gaussian (LQG) approach is proposed and thereby an optimal compensator for the problem considered is designed with measurement-based output feedback. As an illustrative example we would like to consider a benchmark problem of the two-mass-spring system as depicted in Figure 3. The dynamical system of two-mass-spring system in Eq.(1) is given in Table 1 where the state vector $x(t)=[x_1(t), x_2(t), x_3(t), x_4(4)^T$; $x_1(t)$ and $x_3(t)$ denote displacement of the masses with unit(m), and $x_2(t)$ and $x_4(t)$ denote its velocities with unit (m/sec). And the control input $u(t)$ just acts on the mass M. Also we select two measurements of $y_k=[x_{1k}, x_{3k}]^T$. Sampling time is preselected as T=0.25sec and computation time-delay is also known as λ=60% of the sampling time selected. Stochastic excitation to the system are a continuous-time random distur-bance $w_c(t)$, corariance with W_c=1 and a discrete random noise sequence v_k with covariance $V_o=I_2$. Under this circumstance, we obtain the linear transformation G_d via cholesky factorization shown in Eq.(6).

$$G_d = \begin{bmatrix} 4.8927\times10^{-6} & 0 & 0 & 0 & 0 \\ 7.8661\times10^{-4} & 1.2121\times10^{-4} & 0 & 0 & 0 \\ 1.5569\times10^{-3} & 5.9859\times10^{-4} & 1.2735\times10^{-4} & 0 & 0 \\ 9.7134\times10^{-2} & 7.8280\times10^{-2} & 4.7019\times10^{-2} & 1.5823\times10^{-2} & 0 \\ 8.5509\times10^{-2} & 2.4002\times10^{-1} & 4.7905\times10^{-1} & 6.6349\times10^{-1} & 5.1476\times10^{-1} \end{bmatrix}$$

Now, suppose the weighting matrices $Q=I_4$ and $R=1$ in the performance index J_c in Eq.(3). Then the weighting matrices $\bar{Q}$, $\bar{M}$ and $\bar{R}$ in Eq.(10) are yielded which include inter-sample behaviour of the sampled-data system with delayed control action.

$$\bar{Q}=\begin{bmatrix} 1.0000 & 0.0125 & 0.0059 & 0.0000 & 0.0000 & 0.0001 \\ 0.0125 & 1.0002 & 0.6946 & 0.0059 & 0.0002 & 0.0105 \\ 0.0059 & 0.6946 & 78.3140 & -6.8929 & -0.4021 & 0.1047 \end{bmatrix}$$

$$\begin{bmatrix} 0.0000 & 0.0059 & -6.8929 & 0.8787 & 0.0421 & -0.0092 \\ 0.0000 & 0.0002 & -0.4021 & 0.0421 & 0.0023 & -0.0005 \\ 0.0001 & 0.0105 & 0.1047 & -0.0092 & -0.0005 & 0.6003 \end{bmatrix}$$

$$\bar{M} = [0.0000 \quad 0.0020 \quad 0.0282 \quad -0.0017 \quad -0.0001 \quad 0.0001]^T$$

$$\bar{R} = [0.4000]$$

With the performance index $J(u_k, \lambda)$ in Eq.(9), an optimal full state-feedback control-law K^λ in Eq.(14) is obtained.

$$K^\lambda = [-0.9679 \quad -1.7596 \quad -14.5396 \quad 0.6246 \quad 0.0443 \quad -0.0370]$$

Note that the optimal full state-feedback gain is obtained via iteration of Eq.(12) until the Riccati solution gets to the steady-state condition in Eq.(13). Next, let us design an optimal estimator implementing the full state-feedback control-law associated with the measurement y_k. With the random disturbance sequence ξ_k and measurement noise sequence v_k selected, we obtain the optimal estimator gain L_c in Eq.(16) via *iteration* of the Riccati equation in Eq.(17).

$$L_c = \begin{bmatrix} 3.5272 \times 10^{-2} & 2.5346 \times 10^{-2} & -9.7595 \times 10^{-4} & -2.0078 \times 10^{-3} & 3.1897 \times 10^{-6} \\ -9.7595 \times 10^{-4} & 2.5843 \times 10^{-4} & 1.0967 \times 10^{-2} & 2.2732 \times 10^{-3} & 9.7524 \times 10^{-4} \end{bmatrix}^T$$

Once the full state-feedback gain K^λ and the estimator gain L_c are obtained, the optimal compensator taking into account the delayed control action is yielded as given in Eq.(19) which gives optimun cost $J^o(\lambda)=5.3882$.

$$A^\lambda_c = \begin{bmatrix} 9.6407 \times 10^{-1} & 2.4912 \times 10^{-2} & 1.7422 \times 10^{-2} & 1.7783 \times 10^{-4} & 7.1615 \times 10^{-6} \\ -3.3013 \times 10^{-2} & 9.8242 \times 10^{-1} & 1.1843 & 2.3602 \times 10^{-2} & 1.1307 \times 10^{-3} \\ 8.8139 \times 10^{-4} & 8.7518 \times 10^{-5} & 8.0081 \times 10^{-1} & 2.3356 \times 10^{-2} & 1.1376 \times 10^{-3} \\ -3.8664 \times 10^{-3} & 1.7411 \times 10^{-2} & -1.4470 \times 10^{+1} & 8.0284 \times 10^{-1} & 5.4395 \times 10^{-2} \\ -6.2846 \times 10^{-8} & 0 & -1.9215 \times 10^{-5} & 0 & 1.9703 \times 10^{-2} \end{bmatrix}$$

$$B_{z}^{\lambda} = \begin{bmatrix} 3.5885 \times 10^{-2} & -7.8661 \times 10^{-4} & 2.5975 \times 10^{-4} \\ 2.3343 \times 10^{-2} & 1.3453 \times 10^{-2} & 1.4493 \times 10^{-2} \\ -8.3324 \times 10^{-4} & 8.9347 \times 10^{-3} & -2.5078 \times 10^{-4} \\ 1.3444 \times 10^{-2} & -1.5856 \times 10^{-1} & -1.3126 \times 10^{-2} \\ 6.2846 \times 10^{-8} & 1.9215 \times 10^{-5} & 0 \end{bmatrix}$$

$$C_{z}^{\lambda} = [-9.0215 \times 10^{-1} \quad -1.7596 \quad -1.4382 \times 10^{+1} \quad 6.2459 \times 10^{-1} \quad 4.4254 \times 10^{-2}]$$

$$D_{z}^{\lambda} = [-6.5804 \times 10^{-2} \quad -1.5750 \times 10^{-1} \quad -3.6994 \times 10^{-2}]$$

With the optimal compensator, we analyzed the closed-loop sampled-data control system with the delayed control action and showed results in Table 2 and Table 3 where *SDS*1 denotes the design case with the delay time compensated in the LQG design, *SDS*2 the case with the delay time not compensated in design process and *SDS*3 the case without the computation time-delay. And λ_z, ξ_{eq} and ω_n mean eigenvalue, damping ratio and natural frequency, respectively. From results of the cases *SDS*2 and *SDS*3 in Table 2, we see that the computation time-delay may influence seriously on stability robustness and performance of digital control system; especially, flexible modes may be affected by the delayed control action. However, when the time-delay is incorporated in design process, stability robustness and performance of the system with the delayed control action may get recovered to (or even better than) level of the system without the time-delay. In analysis of the control system performance, effect of the incorporation of computation time-delay in design process is apprent to RMS (root-mean-square) responses of the rate states (x_2 and x_4) as shown in Table 3. Furthermore the incorporation of computation time-delay in design process may keep the sampled-data control system out of control saturation. Note that the mode 0 of case *SDS*1 came from need of the delayed control input u_{k-1} to the LQG compensator. Also we note from Table 2 that the closed-loop eigenvalues consist of those of regulator (R) and estimator (E).

VI. Conclusion

In this paper, we treated a practical issue of sampled-data control system; time-delay at control action due to computation time. The problem is formulated in state space and solved entirely in the discrete-time domain. As a design method, we considered the Linear Quardratic Gaussian approach for a measurement-based output feedback problem. From results of the problem, the delayed control action due to the computation time may severely influence on degradation of stability and performance as well as robustness in digital control systems. However, incorporation of the influence in design process may recover (or enhance) level of stability and performance as well as robustness in digital control systems. From the example of two-mass-spring system, flexible modes are easily affected by the time-delay. With the design method incorporating the time-delay, we may keep the sampled-data control system from saturation of control action.

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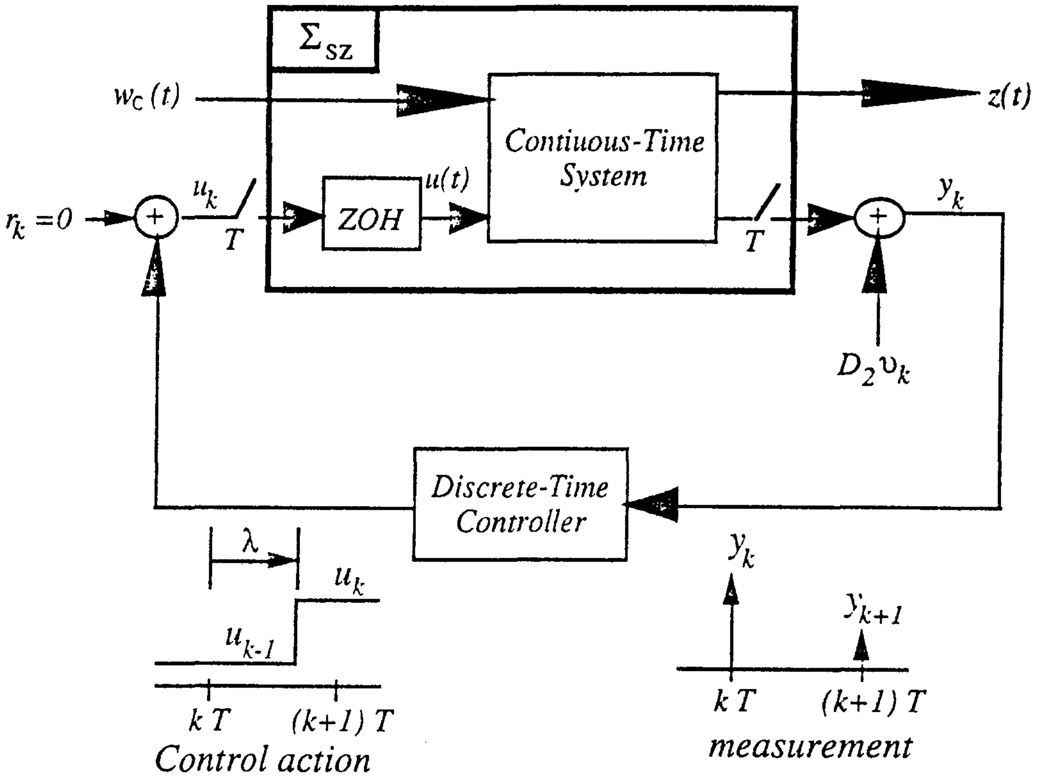

Figure 1: Sampled-Data System with Computation Time-Delay

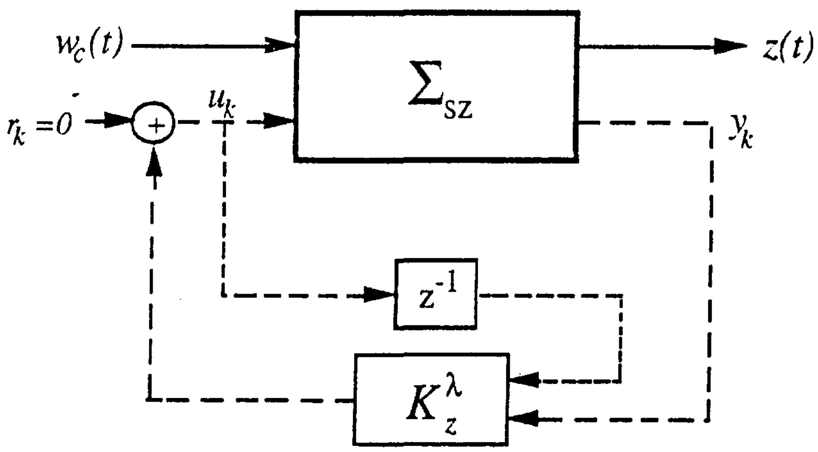

Figure 2: Compensated Closed-Loop System with the Delayed Control Action

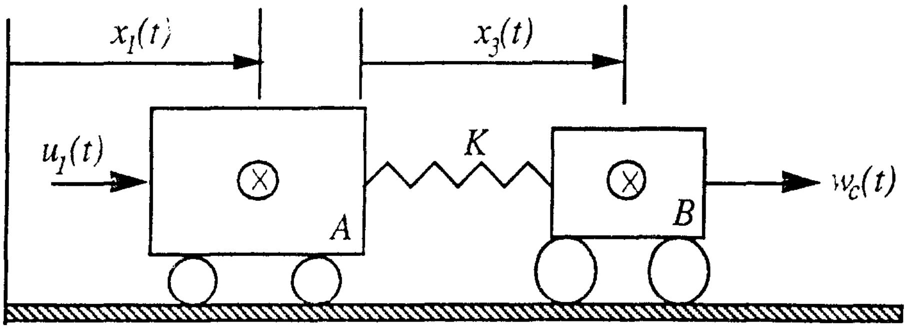

Figure 3: A Two-Mass-Spring System

Table 1: State-Space Model of Two-Mass-Spring System

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 57.423 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -631.655 & 0 & 10 \\ 0 & 0 & 0 & 0 & -157.080 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 17.725 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0.633 & 0 \\ 0 & 0.633 \end{bmatrix}$$

$$D_{12} = 0_{4\times 1}$$

Table 2: Analysis of the Closed-Loop Sampled-Data System

cases	λ_z	$\xi_{eq}^{\dagger}$	$\omega_n(rad/sec)^{\dagger}$	
$SDS1$	$0.79918 \pm 0.58045i$	0.020	25.131	$R^{\ddagger}$
	$0.80452 \pm 0.58455i$	0.009	25.135	$E^{\ddagger}$
	$0.98209 \pm 0.01756i$	0.708	1.012	E
	$0.97980 \pm 0.01220i$	0.853	0.954	R
	0.01970	1.0	157.08	R
	0			R
	0.01970	1.0	157.08	E
$SDS2$	$0.79674 \pm 0.58475i$	0.019	25.330	$R^{\ddagger}$
	$0.80452 \pm 0.58455i$	0.009	25.135	$E^{\ddagger}$
	0.04073	1.0	128.03	R
	$0.98209 \pm 0.01756i$	0.708	1.012	E
	$0.97946 \pm 0.01222i$	0.856	0.9661	R
	0.01970	1.0	157.08	E
	0.01970	1.0	157.08	R
$SDS3$	$0.79918 \pm 0.58045i$	0.020	25.131	$R^{\ddagger}$
	$0.80452 \pm 0.58455i$	0.009	25.135	$E^{\ddagger}$
	$0.98209 \pm 0.01756i$	0.708	1.012	E
	$0.97980 \pm 0.01220i$	0.853	0.9535	R
	0.01970	1.0	157.08	E
	0.01970	1.0	157.08	R

† s−domain equivalent.

‡ R, E mean regualtor and estimator, respectively.

Table 3: RMS responses of States and Control

Variable	$SDS1$	$SDS2$	$SDS3$
x_1	0.2359	0.2343	0.2339
x_2	0.2100	0.2128	0.2095
x_3	0.0672	0.0671	0.0672
x_4	1.6863	1.6865	1.6864
u_1	2.3561	2.3645	2.3602