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A note on the continuity of the joint numerical range

Park, Young-Sik,* Lee, Byung-Soo**
Dept. of
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〈**Abstract**〉

The purpose of this note is to discuss the continuity of the joint numerical ranige from a Banach algebra B(H) into a collection Σ of the closure of convex subsets of a unitary space C^n in the operator topologies.

Joint numerical range의 連續에 관한 研究

朴 榮 植*・李 炳 守**
數 學 科
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〈要　約〉

本 論文에서는 Banach 代數 $B(H)$에서 unitary space C^n의 convex subset들의 closure의 集合 Σ에로의 함수인 the joint numerical range는 operator topologies에 대해서 항상 連續임을 밝혔다.

I. Introduction

The joint numerical range of an operator A, where $A=(A_1, A_2, A_3, \cdots, A_n)$ is an n-tuple of operators on a Hilbert space H, is defined as the set of all complex numbers $(\langle A_1x,x\rangle, \langle A_2x,x\rangle, \langle A_3x,x\rangle, \langle A_nx,x\rangle : x\in H, \|x\|=1)$. In other words, the joint numerical range $W_n(A)$ of an operator A is a function defined on a Banach algebra $B(H)$ whose range consists of convex subsets of the n-dimensional unitary space C^n. This notion of the joint numerical range was first investigated by Halmos ([3], problem 166).

And for a long time much of the knowledge in the single operator carried over to the analogous situations in the case of n-tuple of operators. Our purpose is to discuss the same subject as these.

Hence it would be quite reasonable to try to define what it means for a function of this kind to be continuous. The purpose of this note is to discuss the continuity of the joint numerical range in the operator topologies. We will proceed a discussion of this note on a Banach algebra $B(H)$ of operators on a separable complex Hilbert space H with the scalar product $\langle , \rangle$ and the norm $\|\cdot\|$.

This note consists of three sections including an introduction.

In section II we shall recall definition of Hausdorff metric in a collection of all non-empty compact subsets of C^n and definition of

* 蔚山工科大學 數學科 助敎授(解析學)
** 釜山大學校 師範大學 數學敎育學科 助敎(解析學)

upper and lower semi-continuity of a function from a metric space into a collection of all non-empty compact subsets of C^n. In section Ⅲ we shall discuss the continuity of the joint numerical range.

Ⅱ. Notation and preliminaries

Let H be a separable complex Hilbert space and $B(H)$, the Banach algebra of bounded linear transformations (operators) on H.

Let $A=(A_1, A_2, A_3, \cdots, A_n)$ be an n-tuple of operators on H. The joint numerical range of A is the subset $W_n(A)$ of the n-dimensional unitary space C^n such that $W_n(A)=\{(\langle A_1x, x\rangle, \langle A_2x, x\rangle, \langle A_3x, x\rangle, \langle A_nx, x\rangle) : x\in H, \|x\|=1\}$ [5]. In the case of $n=1$, it is the usual numerical range of an operator. We will denote by Σ the collection of all non-empty compact subsets of C^n equipped with the Hausdorff metric [2]. For a subset K of C^n, we denote $K+(\varepsilon)=z=(z_1, z_2, z_3, \cdots z_n)\in C^n$: $\text{dist}(z, K)<\varepsilon\}$ for any positive number ε.

Lemma 2.1. Let (C^n, d) be a metric space. Define $h(A, B)=\sup\{d(a, B) : a\in A\}$ and $\rho(A, B)=\max[h(A, B), h(B, A)]$ for $(A, B)\in\Sigma\times\Sigma$, then ρ is a metric in Σ.

Definition 2.2[4]. Let (X, e) be a metric space and Σ the collection of all non-empty compact subsets of C^n equipped with the Hausdorff metric. If $f: X\longrightarrow\Sigma$ is a function, then f is said to be upper semi-continuous (lower semi-continuous) at $x\in X$ if for each $\varepsilon>0$, there exists a $\delta>0$ such that $e(x_n, x)<\delta$ implies $f(x_n)\subset f(x)+(\varepsilon)$ (respectively, $f(x)\subset f(x_n)+(\varepsilon)$).

The following lemma is due to Bezak [1].

Lemma 2.3. For a topological space X and a metric space (C^n, d) of finite diameter, let ρ be the Hausdorff metric induced by d on Σ. If f maps X into Σ, then f is upper and lower semi-continuous at $x\in X$ if f is continuous at x with respect to ρ. Conversely, if f is upper and lower semi-continuous at $x\in X$, then f is continuous at x with respect to ρ.

Ⅲ. The continuity of the joint numerical range

Since the Hausdorff metric is defined for compact sets, the appropriate function to discuss is $\text{cl}(W_n)$, the closure of W_n, not W_n.

Lemma 3.1 ([3], problem 167). The joint numerical range of an operator is always convex.

Theorem 3.2. The function $\text{cl}(W_n)$ is continuous with respect to the topologies for operators.

(Proof). If $\|A-B\|<\varepsilon$, and if x is a unit vector, where $A=(A_1, A_2, A_3, \cdots A_n)$, $B=(B_1, B_2, B_3, \cdots B_n)$ then $|\langle(A-B)x, x\rangle|<\varepsilon$.

Since $\langle(A_1, A_2, A_3, \cdots A_n)x, x\rangle=\langle Ax, x\rangle=\langle Bx, x\rangle+\langle(A-B)x, x\rangle=\langle(B_1, B_2, B_3, \cdots, B_n)x, x\rangle+\langle(A_1-B_1, A_2-B_2, A_3-B_3, A_n-B_n)x, x\rangle=\langle(B_1, B_2, B_3, \cdots, B_n)x, x\rangle+(\langle(A_1-B_1)x, x\rangle, \langle(A_2-B_2)x, x\rangle, \langle(A_3-B_3)x, x\rangle, \cdots, \langle(A_n-B_n)x, x\rangle)$, $W_n(A)\subset W_n(B)+(\varepsilon)$. Symmetrically, $W_n(B)\subset W_n(A)+(\varepsilon)$. Hence $\text{cl}(W_n)$ is continuous with respect to the topologies for operators.

Corollary 3.3. The function $\text{cl}(W)$ is continuous with respect to the topologies for operators, $W(A)$ is the numerical range of A.

(Proof). $W(A)=W_1(A)$.

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