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Remarks on Absolutely Flat Rings

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〈Abstract〉

Let A be a commutative, Noetherian and absolutely flat ring with identity. Then A is isomorphic to a direct product of a family of finitely many fields. In general, a commutative and absolutely flat ring with identity is isomorphic to a subdirect product of fields.

Absolutely flat 환에 관하여

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〈요 약〉

A가 absolutely flat이고 항등원을 가진 가환노이더환이면 A는 유한개 체(field)들의 직적(direct product)과 동형이다. 좀 더 일반적으로 A가 absolutely flat이고 항등원을 가진 가환환이면 A는 체들의 직적의 부분환 subdirect product와 동형이다.

I. Introduction

In module theory, flatness is one of the remarkable properties. Precisely, an A-module M is said to be an A-flat module if the functor T_M: $N \to N \otimes_A M$ on the category of A-modules and homomorphisms is exact. More precisely, the following conditions are equivalent.

a) M is an A-flat module.

b) If the A-module homomorphism f: $N' \to N$ is injective, then $f \otimes 1$: $N' \otimes_A M \to N \otimes_A M$ is injective.

By the way, whether an A-module M is flat or not is much concerned with the properties of ring A. For example, if A is a P.I.D., then an A-module M is flat iff M is torsion free. (5) A ring A is called absolutely flat ring if every A-module is flat.

In this paper, we will prove some known conditions equivalent to absolutely-flatness. Using these conditions, we will prove that every commutative, Noetherian and absolutely flat ring with identity is isomorphic to a direct product of a family of finitely many fields. And in general, a commutative and absolutely flat ring with identity is isomorphic to a subdirect product of a family of fields. Throughout this paper, every ring is a commutative ring with identity. First of all, we need following proposition.

Proposition 1.1 Let M be an A-module. Then M is flat if and and only if Tor_1 $(A/\mathfrak{a}, M)=0$ for all finitely generated ideal $\mathfrak{a}$ in A.

Proof See (3).

Proposition 1.2 The following conditions are equivalent. Let A be a ring.

a) A is an absolutely flat ring.

b) Every principal ideal is idempotent, i.e., $(x)=(x^2)$ for all $x \in A$.

c) Every prime ideal of A is a maximal ideal and nilradical of A, denoted by rad (A), is zero.

d) A_m (the localization of A at m) is a field for each maximal ideal of A, m.

Proof Suppose A is absolutely flat. Then, $A/(x)$ and (x) are A-flat modules for all $x \in A$.

Since $0 \to (x) \xrightarrow{i} A \xrightarrow{j} A/(x) \longrightarrow 0$ is exact,

$0 \longrightarrow (x) \otimes_A (x) \xrightarrow{i \otimes 1} A \otimes_A (x) \xrightarrow{j \otimes 1} A/(x) \otimes_A (x) \longrightarrow 0$ is exact. The injectiveness of the map $i \otimes 1$: $(x) \otimes_A A/(x) \to A \otimes_A A/(x)$ and $\mathrm{Im}(i \otimes 1)=0$ implies $(x) \otimes_A A/(x)=0$. Hence $0 \longrightarrow (x) \otimes_A (x) \xrightarrow{i \otimes 1} A \otimes_A (x) \longrightarrow 0$ is exact. And $A \otimes_A (x)$ is isomorphic to (x) by the map $a \otimes bx \longrightarrow abx$. Therefore, $(x)=(x^2)$ for all $x \in A$. Thus a) implies b). Assume b) holds. If $r \in \mathrm{rad}(A)$, then $r^n=0$ for some $n \in Z^+$. By assumption, $(r)=(r^2)=(r^4)\cdots=(r^{2n})=0$. This shows $\mathrm{rad}(A)=0$. Let p be a prime ideal in A and $a \notin p$. Then $a=a^2x$ for some $x \in A$, Hence $a(1-ax)=0$ and we have $1-ax \in p$. Whence $(a)+p=A$. This shows that p is maximal. To prove that c) implies d), we show that A_m has only zero-ideal. Since $S^{-1}m$ is the only prime ideal of A, where S means the multiplicative set A-m, $\mathrm{rad}(A_m)=S^{-1}m$. For $[a/s] \in S^{-1}m$, there exists $n \in Z^+$ such that $[a/s]^n=0$.

This means there exist $r \in S$ such that $a^n r=0$ in A. Therefore, $(ar)^n=0$ in A. This implies $ar=0$ since $\mathrm{rad}(A)=0$. Hence $[a/s]=0$. So A_m has only zero ideal. Assume d) holds. Since A_m is a field, the only ideals in A_m are 0 and itself. Hence $\mathrm{Tor}_1(A_m/a, M)=0$ for all A_m-modules, M and all ideals of A_m, a. (5) By proposition 1.1, every A_m-module is flat. From this fact, it follows that every A-module is flat. (1) Q.E.D.

Ⅱ. The structure theorems of absolutely flat rings

To prove the structure theorems, we need some known knowledge and definitions.

Definition **2.1** A ring $A \neq 0$ is said to have dimension zero if all prime ideals are maximal. (2)

Definition **2.2** Let $\{A_i\}_{i \in I}$ be a family of rings indexed by some set I. The direct product of the rings A_i, denoted by $\prod_{i \in I} A_i$, consists of all functions α defined on the index set I subject to the conditions that for each element $i \in I$, $\alpha(i) \in A_i$. $\prod_{i \in I} A_i = \{\alpha \mid \alpha: I \to \bigcup_{i \in I} A_i;\ \alpha(i) \in A_i\}$ A subring S of $\prod_{i \in I} A_i$ is said to be a subdirect product of the rings A_i, written as $\prod^s_{i \in I} A_i$, if the induced projections $\prod_i S$: $S \to A_i$ is an onto mapping for each i.

Theorem **2. 1** A ring $A \neq 0$ is Artinian (2) if and only if it is Noetherian (2) and of dimension zero.

Proof See (2).

Theorem **2. 2** If a ring A is Artinian, then A has only finite maximal ideals.

Proof Suppose that A has infinitely many maximal ideals. Choosing infinitely countable maximal ideals $\{m_i\}_{i \in Z^+}$, we consider the following discending chain of ideals. $m_1 \supset m_1 m_2 \supset m_1 m_2 m_3 \supset \cdots\cdots$

This is a strictly discending chain of ideals and hence contradicts to the fact that A is Artinian.

Theorem **2.3** A ring A is Noetherian and absolutely flat if and only if A is isomorphic to a direct product of a family of finitely many fields.

Proof Suppose A is Noetherian and absolutely flat. By theorem 2.1 and theorem 2.2, A has only finite maximal ideals $m_1, m_2, \cdots\cdots, m_n$. By the condition c) of proposition 1.2, $\bigcap_{i=1}^{n} m_i = \mathrm{rad}(A)=0$. Appealing to Chinese remainder theorem (4), we have A is isomorphic to $\prod_{i=1}^{n} A/m_i$, where every A/m_i is a field. The converse is obvious by condition b) of proposition 1.2.

***Theorem* 2.4** An absolutely flat ring is isomorphic to a subdirect product of fields.

Proof Let $\{m_i\}_{i\in I}$ be the family of all maximal ideals of an absolutely flat ring A. Define a function f: $A\rightarrow\prod_{i\in I}A/m_i$, by requiring $f(a)$ to be such that its projection $\prod_i(f(a))=a+m_i$.

Then ker f

$$=\{a\in A\,|\,(\prod_i\cdot f)(a)=m_i \text{ for all } i\in I\}$$
$$=\{a\in A\ a+m_i=m_i \text{ for all } i\in I\}$$
$$=\bigcap_{i\in I}m_i$$
$$=\text{rad}(A)=0 \text{ (by proposition 1.2)}$$

This implies A is isomorphic to $f(A)$, a subring of $\prod_{i\in I}A/m_i$. And since $\prod_i\cdot f(A)=A/m_i$, *i.e.*, $\prod_i$ is an onto mapping for each $i\in I$, A is isomorphic to $\prod'_{i\in I}A/m_i$. Q.E.D.

Remark The converse of theorem 2.4 is not true. While $Z\cong\prod' Z_{p_i}$, where p_i runs all prime numbers, Z is not absolutely flat.

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