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Quasi Semiopen sets

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〈**Abstract**〉

The object of this paper is to introduce a quasi semiopen set and some newtype of separation axioms. We will investigate their characterizations.

준 반 개집합에 관하여

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〈**요 약**〉

이 논문에서 준 반 개집합을 도입하고 또 새로운 분리공리를 도입하여 그들의 성질을 알아본다.

In[5], the concept of pairwise semi T_1 spaces has been introduced. The problem arises as to what is the nature of singleton sets in these spaces. While studying this problem we are led to define quasi semiopen sets. These sest are utilized and some new type of separation axioms are introduced and its basic properties are obtained.

I. Preliminaries

Definition A. In a topological space X a set A is semi-open if for some open set O, $O \subset A \subset$ (closure of O in X)[3].

Every open set is semiopen but not conversely[3]. Complement of a semi open set is semi closed[1].

Theorem A. In a topological space any union of semi open sets is semi open[3]

Theorem B. Let Y be a subspace of a topological space X and $A \subset Y$. If A is semi open in X then A is semi open in Y[3].

Theorem C. In a topological space intersection of an open set and a semiopen set is semi open[4].

Theorem D. Let Y be a subspace of a topological space X. If A is semiopen in Y and Y is semi open in X, then A is semi open in X[4].

Definitions B. In a topological space X a point p is said to be a semi limit point of A if each semi open set containing p contains a point of A distinct from p. The set of all the semi limit points of A is denoted by $sd(A)$. The union of A and $sd(A)$ is denoted by scl$A$[1].

Definition C. A bitopological space(X, P, Q) is a non empty set X equipped with two topological P and Q[2].

Definition D. A space(X, P, Q) is pairwise semi T_0(resp. pairwise semi T_1) if for $x, y \in X$, $(x \neq y)$ there is a p-semi open set containing x but

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not y or(resp. and) a Q-semi open set containing y but not x[5].

Definition E. A space(X,P,Q) is pairwise semi T_2 if for $x,y\in X$, $x\neq y$, there exists a p-semi opem set U and a Q-semi open set V such that $x\in U$ $y\in V$ and $U\cap V=\phi$[5].

Pairwise semi T_2 implies pairwise semi T_1 and pairwise semi T_1 implies pairwise semi T_0 but the converses of these implications need not be true[5].

Throughout this note a space(X,P,Q) stands for a bitopological space, P-semi open means semiopen relative to P and $X-A$ denotes the complement of A in X.

II Quasi semi open sets

Definition 1. In a space(X,P,Q) a set A is termed quasi semi open(written QSO) if for $x\in A$ there exists either a P-semi open set U such that $x\in U\subset A$ or a Q-semi open set V such that $x\in V\subset A$.

Remark 1. Every P-semi open(resp. Q-semi open) set is QSO. The converse may be false.

Example 1. Let $X=\{a,b,c\}$, $P=\{\emptyset, \{a\}, \{b,c\}, X\}$ and $Q=\{\emptyset, \{b\}, \{a,c\}, X\}$. Then $\{a,b\}$ is QSO but it is neither P-semi open nor Q-semi open.

Theorem 1. A is QSO in a space(X, P, Q) if and only if it is a union of P-semi open set and a Q-semi open set.

Proof. *Necessity*: Let $x\in A$. Then there exists either a P-semi open set U_x such that $x\in U_x\subset A$ or a Q-semi open set V_x such that $x\in V_x\subset A$. Let A_1 be the subset of A such that for every $x\in A_1$ there is a P-semiopen set U_x such that $x\in U_x\subseteq A$. Then $A-A_1$ is a subset of A such that for every $y\in A-A_1$, there is a Q-semi open set V_y such that $y\in V_y\subset A$. Therefore $A=(\bigcup_{x\in A_1} U_x)\cup(\bigcup_{y\in A-A_1} V_y)$. The result now follows by Theorem A.

The sufficiency is obvious.

Theorem 2. Any union of QSO sets is QSO.

This follows from Theorem 1 and Theorem A.

Remark 2. Intersection of two QSO sets may not be QSO.

Example 2. Let $X=\{a,b,c\}$, $P=\{\emptyset, \{a\}, \{b,c\}, X\}$ and $Q=\{\emptyset, \{a\}, X\}$. Then the sets $\{b,c\}$, $\{a,b\}$ are QSO but their intersection $\{b\}$ is not QSO. It is clear that the intersection of even a P-open set and QSO set may not be QSO. However.

Theorem 3. Intersection of a biopen set and a QSO set is QSO.

Proof. Let O be biopen and A be a QSO set in a space(X,P,Q). There is a P-semi open set W and Q-semi open set V such that $A=W\cup V$. And so, $O\cap A=(O\cap W)\cup(O\cap V)$. Now O being biopen(i.e. both P-open and Q-open). We get by theorem C that $O\cap W$ is P-smi open and $O\cap V$ is Q-semi open. Hence by Theorem 1, $O\cap A$ is QSO.

Theorem 4. Let $A\subset Y\subset X$ and (Y, P_Y, Q_y) be a subspace of (X,P,Q). If A is QSO in X then it is QSO in Y.

This follows from Theorem 1 and Theorem 3.

Theorem 5. Let (Y, P_Y, Q_Y) be a subspace of space (X,P,Q). If A is QSO in X and Y is biopen in X, then $A\cap Y$ is QSO in Y.

This follows from Theorems 3 and 4.

Theorem 6. Let (Y, P_Y, Q_Y) be a subspace of a space (X, P, Q). If A is QSO in (Y, P_Y, Q_Y) and Y is biopen in (X,P,Q) then A is QSO in X.

This follows from Theorem 1 and Theorem D.

Definition 2. In a space (X,P,Q) a set A is quasi semiclosed (written, QSC) if its complement is QSO.

Remark 3. Every P-semi closed(resp. Q-semi closed) set is QSC. The converse may be false.

Theorem 7. Any intersection of QSC sets is QSC.

Remark 4. Union of two QSC sets may not be QSC.

Definition 3. In a space (X,P,Q) a point $p\in X$ is termed quasi semi limit point of A if for any QSO set O containing p, $O\cap(A-\{p\})\neq\phi$. Denote the set of all the quasi semi limit points of A by $qsd(A)$.

Theorem 8. Let A be a subset of a space (X,P,Q). Then, $qsdA=P\text{-}sd(A)\cap Q\text{-}sd(A)$.

Proof. By Remark 1, $qsd(A)\subset P\text{-}sd(A)\cap Q\text{-}sd(A)$. Conversely let $p\in p\text{-}sd(A)\cap Q\text{-}sd(A)$. Since $p\in P\text{-}sd(A)$, every P-semi open set which contains p contains a point of A other than p. Let O be any QSO set such that $p\in O$. By Theorem 1 there exists a p-semi open set W and a Q-semiopen set V such that $O=W\cup V$. Therefore $O\cap A=(A\cap W)\cup(A\cap V)$. Consequently, $O\cap A$ contains a point of A distinct from p. Hence $p\in qsd(A)$.

Theorem 9. In a space (X,P,Q) a set A is QSC iff $qsd(A)\subset(A)$.

Proof. *Necessity*: Let $p\in qsd(A)$. Assume that $p\notin A$. Then $p\in X-A$. Since $X-A$ is QSC and disjoint from A, $p\notin qsd(A)$. This is a contradiction. So, $p\in A$. Therefore $qsd(A)\subset A$.

Sufficiency: Suppose that $qsd(A)\subset A$. Let $p\in X-A$. Then $p\notin A$. So, $p\notin qsd(A)$. Hence there is a QSO set O which contains p but contains no point of A. Therefore, $p\in O\subset X-A$. And so $X-A$ is a union of QSO sets which by Theorem 2 is QSO. Hence A is QSC.

Theorem 10. If $A\subset B$ then $qsd(A)\subset qsd(B)$. This follows from Definition B.

Theorem 11. $(qsd\ qsd(A))-A\subset qsd(A)$.

Proof. Let $p\in(qsd\ qsd(A))-A$ and O be a QSO set containing p. Then $O\cap(qsd(A)-\{p\})\neq\phi$. Let $a\in O\cap(qsd(A)-\{p\})$. Now since $a\in qsd(A)$ and $a\in O$, so $O\cap(A-\{a\})\neq\phi$. Let $r\in O\cap(A-\{a\})$. The $r\neq p$, for $r\in A$ but $p\notin A$. Therefore, $O\cap(A-\{p\})\neq\phi$. Consequently $p\in qsd\ (A.)$

Theorem 12. $qsd(A\cup qsd(A))\subset A\cup qsd(A)$.

Proof. Let $p\in qsd\ (A\cup qsd(A))$. If $p\in A$, then trivial. So let $p\notin A$. Now, if O is a QSO set contains p then $O\cap[(A\cup qsd(A))-\{p\}]\neq\phi$. This implies that $O\cap(A-\{p\})\neq\phi$. For, otherwise $O\cap[qsd(A)-\{p\}]\neq\phi$. And so by the proof in Theorem 11, $O\cap(A-\{p\})\neq\phi$. Hence $p\in qsd(A)$.

Definition 4. In a space (X,P,Q), the set $A\cup qsd(A)$ is termed the quasi semi closure of A. Denote it by $qscl(A)$.

Theorem 13. In a space (X,P,Q) we have

(a) $qscl(A)=P\text{-}scl(A)\cap Q\text{-}scl(A)$. [by Th. 5].

(b) If $A\subset B$ then $qscl(A)\subset qscl(B)$ [by Th. 10]

(c) $qscl(qscl(A))=qscl(A)$ [by Th. 12]

(d) A is QSC iff $qscl(A)=A$ [by Th. 9]

(e) $qscl(A)$ is QSC [by (c) and (d)].

(f) $qscl(A)=\cap\{F|F$ is QSC, $A\subset F\}$ [by(*b*), (*d*), (*e*)].

(g) $qscl(A)$ is the smallest QSC set which contains A. [by(*f*)].

(h) $x\in qscl(A)$ iff each QSO set containing x meets A [by def. 4].

Theorem 14. If (Y, P_Y, Q_Y) is a biopen subspace of a space (X,P,Q) then for any subset B of Y, $qscl_Y(B)=qscl(B)\cap Y$, where $qscl_Y(B)$ denotes the quasi semi closure of B in the subspace(Y, P_Y, Q_Y).

Proof. Let $x\in qscl_Y(B)$. Then $x\in Y$. Let V be any QSO set in X containing x. Since Y is biopen, $V\cap Y$ is QSO in Y by Theorem 5. So by Theorem 13(*h*), $(V\cap Y)\cap B\neq\emptyset$. Consequently, V meets B. Hence applying Theorem 13(*h*), $x\in qscl(B)$. Thus, $x\in qscl\ (B)\cap Y$.

Conversely, let $y\in qscl(B)\cap Y$ and let O be a QSO set in Y such that $y\in O$. Since Y is biopen O is QSO in X by Theorem 6. And so $O\cap B\neq\emptyset$. Therefore $y\in qscl_Y(B)$.

Ⅲ. Some New Separation Axioms.

Definition 5. A space (X, P, Q) is termed quasi semi T_0 if for each pair of distinct points x, y of X there is a QSO set containing x or y but not the other.

Theorem 15. A space (X, P, Q) is quasi semi T_0 iff it is pairwise semi T_0.

Proof. *Necessity.* Suppose that (X, P, Q) is quasi semi T_0 and let x, y be two distinct points of X. Without any loss in the generality let O be a QSO such That $x \in O$. but $y \notin O$. By Theorem 1 there exist a P-semi open set W and a Q-semi open set V such that $O = W \cap V$. Therefore $x \in W$ or $x \in V$ but $y \notin W$, $y \notin V$. Consequently (X, P, Q) is pairwise semi T_0.

The sufficiency follows for every P-semiopen (resp. Q-semi open) set is QSO.

Theorem 16. A space(X, P, Q) is quasi semi T_0 iff for any two distinct points x, y of X, $\text{qscl}\{x\} \neq \text{qscl}\{y\}$.

Proof. *Necessity*: Let $x, y \in X$. $x \neq y$. without any loss in the generality assume that O is a QSO set containing x but not y. Then $y \in \text{qscl}\{y\} \subset X - O$ And so $x \notin \text{qscl}\{y\}$. Since $x \in \text{qscl}\{x\}$ it follows that $\text{qscl}\{x\} \neq \text{qscl}\{y\}$.

Sufficiency: Let $x, y \in X$, $x \neq y$ and $\text{qscl}\{x\} \neq \text{qscl}\{y\}$. And so let $p \notin \text{qscl}\{x\}$ but $p \in \text{qscl}\{y\}$. Then $y \notin \text{qscl}\{x\}$. For if $y \in \text{qscl}\{x\}$, then $\text{qscl}\{y\} \subset \text{qscl}(\text{qscl}\{x\}) = \text{qscl}\{x\}$. And so $p \in \text{qscl}\{x\}$, a contradiction. Therefore $X - \text{qscl}\{x\}$ is a QSO set containing y but not x. Hence (X, P, Q) is quasi semi T_0.

Definition 6. A space (X, P, Q) is termed quasi semi T_1 if for each pair of distinct points x, y of X there exist a QSO set containing x but not y and a QSO set containing y but not x.

Theorem 17. Every quasi semi T_1 space is quasi semi T_0

The converse may be false. For,

Example 3. Let $X = \{a, b, c\}$, $P = \{\emptyset, \{a\}, X\}$ and $Q = \{\emptyset, X\}$. Then the space(X, P, Q) is quasi semi T_0 but it is not quasi semi T_1.

Theorem 18. Every pairwise semi T_1 space is quasi semi T_1.

The converse may be false. For,

Example 4. Let $X = \{a, b, c\}$. $P = \{Q, \{a\}, \{b\}, \{a, b\}, X\}$. and $Q = \{\emptyset, \{a\}, \{b, c\}, X\}$ Then the space(X, P, Q) is quasi semi T_1 but it is not pairwise semi T_1.

Theorem 19. A space (X, P, Q) is quasi semi T_1 iff the singletons are QSC.

Proof. *Necessity.* Let(X, P, Q) be quasi semi T_1 and $x \in X$, For $y \in X$, $y \neq x$, there exists a QSO set O containing y but not x. Then $X - O$ is QSO set containing x but not y. So the intersection of all the QSC sets containing x does not contain any point other than x. Consequently $\{x\}$ is QSC. The sufficiency is obvious.

Theorem 20. In a pairwise semi T_1 space all the singletons are QSC.

This follows form Theorems 18 and 19.

Remark 5. Example 4 shows that not even a biclosed subspace of a quasi semi T_1 space is quasi semi T_1. However,

Theorem 21. Every biopen subspace of a quasi semi T_1 space is quasi semi T_1.

Definition 7. A space (X, P, Q) is termed quasi semi T_2 if for each pair of distinct points x, y of X there exist QSO sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 22. Every pairwise semi T_2 space is quasi semi T_2.

Remark 6. The space (X, P, Q) of example 4 is quasi semi T_2 but it is not pairwise semi T_2.

Theorem 23. Every quasi semi T_2 space is quasi semi T_1.

The converse may be false. For,

Example 5. Let $X = \{a, b, c\}$, $P = \{\emptyset, \{a\}, X\}$ and $Q = \{\emptyset, \{b, c\}, X\}$. Then the space$(X, P, Q)$ is quasi semi T_1 but it is not quasi semi T_2.

Remark. 7. The space (X, P, Q) of Example (4) is quasi semi T_2 but it is not pairwise semi T_1. On the other hand if X be an infinite set and $P = Q =$ cofinite topology on X then every P-open set intersects every Q-open set. This implies that every P-semiopen set meets every Q-semi open set. Consequently any two quasi semi open sets meets. For if U and V be any two QSO sets then there are P-semiopen sets A, A_1 and Q-semi open sets B, B_1 such that $U = A \cup B$ and $V = A_1 \cup B_1$. It follows that $U \cap V = (A \cup B) \cap (A_1 \cup B_1) = (A \cap A_1) \cup (B \cap A_1) \cup (A \cap B_1) \cup (B \cap B_1) \neq \emptyset$. Hence, the space (X, P, Q) is not quasi semi T_2. Evidently, it is pairwise T_1

and consequently pairwise semi T_1.

Theorem 24. In a space (X, P, Q) the following statements are equivalent.

(a) (X, P, Q) is quasi semi T_2

(b) Let $x \in X$. For each point y distinct from x there exists a QSO set O such that $x \in O$ and $y \in$ qscl O.

(c) For each $x \in X$, $\{x\} = \cap \{\text{qscl } O \mid x \in O,\ O \text{ is QSO}\}$.

The road map for the proof is (a)$\Longrightarrow$(b)$\Longrightarrow$(c). Being straight forward it is left to the reader.

Remark 8. Not even a biclosed subspace of a quasi semi T_2 space is quasi semi T_2. For in example 4, $\{b, c\}$ is a biclosed subspace of the quasi semi T_2 space but it is not quasi semi T_2. However,

Theorem 25. Every biopen subspace of a quasi semi T_2 space is quasi semi T_2.

The following diagram shows the implications between separation axioms appeared in this note,

$$\begin{array}{ccc} \text{pairwise semi } T_2 & \Longrightarrow & \text{quasi semi } T_2 \\ \Downarrow & & \Downarrow \\ \text{pairwise semi } T_1 & \Longrightarrow & \text{quasi semi } T_1 \\ \Downarrow & & \Downarrow \\ \text{pairwise semi } T_0 & \Longleftrightarrow & \text{quasi semi } T_0 \end{array}$$

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