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Semisimplicity of Fixed Jordan Subrings of a Group of Jordan Automorphisms of a Ring R^*.

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〈Abstract〉

Let R be an associative ring and G be a group of some Jordan automorhisms of R. The semisimplicit of the fixed jordan subrings of G implies that R is semisimple where R is semiprime and right noetherian and $|G|$ is a bijection on R.

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〈요 약〉

환 R의 Jordan Automorphisms군에 대한 고정 Jordan 부분환이 반단순환일때 환 R도 반 단순환이 됨을 보였다.

1. Irtroduction.

The relations between the structure of R^G and the structure of R were studied by some mathematicians for several years.

Especially these topics were related to the case of ordinary ring automorphisms of R or the case when R has an involution. I. N. Herstein also studied the structure of Jordan ring R^+ with the structure of a ring$R$.

We now explain our terminologies.

(1) If A is an additive subgroup of R, A is a Jordan subring of R if A is closed under squares (that is $x^2 \in A$) and under the quadratic operation where $a \circ b = bab$. In fact if $2R=R$ this definition is equivalent to A being closed under the single linear operation $a \circ b = 1/2(ab+ba)$. For example the ring R itself is a Jordan subring of R. In this case we will denote it by R^+

(2) A mapping $\phi: R \to R'$ of the rings R and R' is a Jordan homomorphism if (i) $\phi(a+b) = \phi(a)+\phi(b)$ (ii) $\phi(a^2) = \phi(a)^2$ (iii) $\phi(bab) = \phi(b)\phi(a)\phi(b)$ for arbitrary a and b in R.

Clearly a ring homomorphism is a Jordan homomorphism.

A Jordan automorphism of R is simply a Jordan homomorphism which is also one to one and onto; we let $\mathrm{Aut}_J(R)$ denote the group of all Jordan automorphisms of R.

(3) $R^G = \{r \in R \mid r^\phi = r$ for every ϕ in $G\}$ is clearly a Jordan subring of R where G is a subgroup

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of $\mathrm{Aut}_J(R)$. We know that R^G is not empty for $0 \in R^G$. Moreover, if G is finite, we define the trace of x by $tr(x) = \sum_{\phi \in G} x^\phi$. Then $tr(x) \in R^G$. We let $tr(R) = \{tr(x) \mid x \in R\}$

We will show some examples.

Example 1.1. Let $M_n(R)$ be the ring of n by n matrices where R is *a* commutative ring. Then $G = \{id, T_t\}$ is a subgroup of $Aut_J(M_n(R))$ where $T_t: A \to {}^tA$. In this case ${}^tA$ is the transpose of A, Thus $M_n(R)^G$ is the set of all symmetric matrices of $M_n(R)$.

Example 1.2. Let R be *a* non-commutative ring with involutions Define $\Psi: R \oplus R \to R \oplus R$ by $\Psi(a, b) = (a^*, b)$, then $\Psi$is *a* Jordan automorphism. Let $G = \{id, \Psi\}$. Then $(R \oplus R)^G = S_R \oplus R$ where $S_R = \{a \in R \mid a^* = a\}$

2. Some basic results.

In this section we studied some results for our main theorem. The following theorem of I. N. Herstein and the corollary of Martindale-Montgomery are basic on our thesis.

Proposition 2.1. (Herstein) Let $\phi: R \to R'$ be *a* Joran homomorphism of R onto *a* prime ring R'. Then ϕ is either a homomorphism or an anti-homomorpsm.

Proof See(3).

In that theorem the hypothesis that R is a prime ring is essential. In example 1.2., Ψ is neither a homomorphism nor an anti-homomorphism. Of course $R \oplus R$ is not prime but semi-prime.

Corollary 2.2. (Martindale-Montgomery) Let ϕ be *a* Jordan automorprhism of R and let P be *a* prime ideal of R. Then P^ϕ is a prime ideal of R. Moreover the prime rings R/P and R/P^ϕ are either isomorphic or anti-isomorphic.

Proof. See(4).

We consider the following example.

Example 2.3. Let T be a simple, non-commutative ring with involution*, and let R be the direct sum of T_i where $T_i = T$ and $1 \leq i \leq n$. Define $\Psi: R \to R$ by $\Psi(a_1, a_2, \cdots a_n) = (a_n^*, a_1 \cdots a_{n-1})$ Then Ψ is neither a homomorphism nor an anti-homomorphism. On the other hand let $P = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-1}$, then P is prime ideal of R and R/P is anti-isomorphic to R/P^ϕ via $(x+P)^\phi = x^\phi + P$ for every $x + P \in R/P$.

The following terminologies are basic for our theorem.

(1) Let A be *a* Jordan subring of R. The additive subgroup $I \subset A$ is said to be a Jordan ideal of A if whenever $b \in I$ and $a \in A$ then $b \circ a \in A$; that is $aba \in I$. Thus every ideal of R is *a* Jordan ideal. But one-sided ideal may be not a Jordan ideal.

(2) If I is an ideal of R, we say that I is G-invariant if $I^\phi \subset I$ for every ϕ in G. In this case G is *a* subgroup of Jordan automorphisms of I via restriction of ϕ on I.

(3) The ring R is said to have no n-torsion (or n torsion free) if $nr = 0$ for some r in R implies $r = 0$.

(4) For some positive integer n, n is *a* bijection on R if (i) $nR = R$ (ii) R has no n-torsion: that is n is *a* bijective function on R.

We remark the followings when G is of finite order n and n is a bijection on R.

(1) $R^G = tr(R)$

Proof. For arbitrary x in R^G there exist some y in R such that $ny = x$. We can denote y by $1/nx$. Then $y^\phi = ((1/n)x)^\phi = (1/n)n((1/n)x)^\phi = 1/n(n(1/n)x)^\phi = (1/n)x^\phi = y$. Thus $x = \sum_{\phi \in G} y^\phi = n((1/n)x) \in tr(R)$. For other direction, we already know that $tr(R) \subset R^G$. Thus $R^G = tr(R)$. Moreover, we know that $nR^G = R^G$.

(2) When$I$ is G-invariant, $\bar{R} = R/I$ has an induced group of Jordan automorphisms, given as follows: for $\phi \in G$, define by $(x+I)^{\bar\phi} = x^\phi + I$. Let K be the kernel of the mapping $\phi \to \bar\phi$ and let $\bar G = G/K$. Then $\bar G$ is *a* group of Jordan automorphisms of R/I. In this case we get $\bar R^{\bar G} = \overline{R^G}$.

Proof. Cleary we have that $\bar{R}^{\bar{G}} \subset \bar{\bar{R}}^{\bar{G}}$ for $(x+I)^{\phi}=x^{\phi}+I=x+I$ for every x in R^{G}. On the other hand if $\bar{x} \in \bar{R}^{\bar{G}}$ then $n\bar{x}=|K||G|x=|K| \sum_{\phi \in G} \bar{x}^{\phi} = \sum_{\phi \in G} |K| \bar{x}^{\phi} = \sum_{\phi \in G} x^{\phi} = tr(x) \in \bar{R}^{G}$. Since $\bar{R}^{\bar{G}} = n\bar{R}^{\bar{G}}$, $\bar{\bar{R}}^{\bar{G}} \subset \overline{R^{G}}$

3. Semisimplicity

In this section we assume that $R=2R$ and $|G|$ is *a* bijection on R.

In a Jordan ring A, the Jacobson radical $J(A)$ is defined as the maximal quasi-regular ideal, where an element $x \in A$ is quasi-regualr if $1-x$ is invertible (if $1 \notin A$, the inverse is formal). When A is a special Jordan ring, say $A \subset R^{+}$, where R is an associative ring, then being ivertible in the Jordan sense is the same as being invertible in the associative sense. Thus x is quasi-regular in A if and only if there exists $y \in A$ such that $x+y+1/2(xy+yx)=0$. We also denote the Jacobson radical of R by $J(R)$: since $J(R)=J(R^{+})$ by a theorem of McCrimmon(5).

To obtain our main results we need the following propositions by Martindale-Montgomery.

Proposition 3.1. If G is *a* finite group of Jordan automorphism of *a* ring R, such that R has no $|G|$-torsion. Then $P(R \cap R^{G}) = P(R^{G})$ where $P(R)$ is the prime radical of R.

Proof. See(4).

Proposition 3.2. Under same finite group G in proposition 3.1. If $|G|$ is a bijectio on R, then $J(R^{G})=J(R) \cap R^{G}$.

Proof. See(4).

From two propostions we know that if R is semi-simple then R^{G} is semisimple and if R is semiprime then R^{G} is semiprime. But the fact that if R^{G} is semisimple then R is semisimple is not known. Here we can prove that.

Lemma 3.3. If R^{G} is nilpotent, then R is a nil ring.

Proof. It is sufficient to show that R has no prime ideals: that is $R=P(R)$. Assume that P is a proper prime ideal of R. If P is G-invariant, then G acts on $\bar{R}=R/P$ by remark (2) and $\bar{R}^{\bar{G}}=\bar{R}^{G}$. On the other hand by proposition 3.1, we know that $P(R^{G})=P(R) \cap R^{G}=\{0\}$ for $P(R)=\{0\}$ (since $\bar{R}$ is prime.) In this case $\bar{R}^{G}$ has no nilpotent ideals. But the fact that R^{G} is nilpotent implies $\bar{R}^{G}$ is nilpotent because $\bar{R}^{G} = \bar{R}^{G}$. Thus P is not G-invariant.

We let $J=\bigcap_{\phi \in G} P^{\phi}$. If $J=\{0\}$ then $P(R)=\{0\} = 0$ implies R^{G} is no nilpotent. If $J \neq \{0\}$ let $R=R/J$ then we know that R is semiprime since all prime ideals of R are of the form P^{ϕ}/J. It is also contradiction. For, also the fact that R^{G} is nilpotent implies that $\bar{R}^{G}$ is nilpotent. But this is impossible because $\bar{R}$ is semiprime.

Theorem 3.4. If R is semiprime and right noetherian. Then if R^{G} is semisimple then R is semisimple.

Proof. It is sufficient to show that $J(R)=\{0\}$. Assume that $J(R) \neq \{0\}$. For arbitrary x in $J(R)$, x^{ϕ} is also in $J(R)$ for every $\phi \in G$ (in fact $x+y+1/2(xy+yx)=0$ implies $x^{\phi}+y^{\phi}+1/2(x^{\phi}y^{\phi}+y^{\phi}x^{\phi})=0$ for every Jordan automorphism) Thus $J(R)$ is G-invariant since $J(R)$ is invariant under any Jordan automorphism of R. We recall that G is *a* group of Jordan automorphisms of $J(R)$ by remark(2). The assumption that $J(R) \neq \{0\}$ implies $J(R) \cap R^{G} \neq \{0\}$ implies $J(R)^{G}=\{0\}$. And if $J(R)^{G}=\{0\}$, then $J(R)$ is nil subring of R by lemma 3.3. But since every nil ideal of right noetherian ring is nilpotent (2), R contains nontrivial nilpotent ideal $J(R)$ (we recall that $J(R)=J(R^{+})$.) This is impossible. Thus we obtains $J(R)=\{0\}$.

Finally we will show that some examples for appropriateness of our theorem; that is there exist many rings which are seiprime and right noetherian but not semisimple.

Example 3.5. Let R be the ring of all rational numbers whose denominators are odd. Then R is commutative prime ring for R has no zero divisors except 0. If q/p is contained in an ideal of R, then q is even number for otherwise 1 is contained in that ideal. Thus the ideal of all rational numbers whose numerators are even is unieue maximal ideal of R i.e. $J(R)\neq 0$. And clealy R is neotherian for every proper ideal is contained in finitely many ideals of R.

This example shows that semiprime noetherian ring may not be semisimple.

4. Questions

If in theorem 3.4. The assumption that R is right noetherian is deleted, is the theorem true? In the course of proof we can know that $J(R)$ is nilpotent since R is right noetherian. In this case if the nilpotency of R^G implies the nilpotency of R, we can delet the assumption that R is right noetherian since $J(R)$ is nilpotent. In fact if G is a group of automorphisms of R and R has no $|G|$-torsion, Then the nilpotency of R^G implies the nilpotency of R by Bergman and Issac. But in case Jordan automorphism group, that has been neither proved nor disproved.

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