울산대학교 연구논문집 제21권(자연과학 · 공학편) 제1호 pp. 111~113, 1990.
UOU Report Vol.21, (Natural Science and Engineeing) No.1, pp. 111~113, 1990.
University of Ulsan

폐그래프와 Hyper 폐함수에 대하여

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〈요 약〉

이 논문에서는 함수가 폐그래프를 갖기위한 필요충분조건을 토대로하여 일반화된 새로운 개념, hyper 폐함수를 도입하고 정의공간, 치역공간등의 성질에 따라 그 새로운 개념이 어떠한 성질들을 갖는가를 조사 연구하였다.

CLOSED GRAPH AND HYPER CLOSED FUNCTIONS

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〈Abstract〉

This article begins with the generalization of necessary and sufficient condition for any function to have closed graph [4]. Its generalization led us introduce a new concept of hyper closed functions. Some propositions have been worked out under different situations making functions hyper closed. We close our article by investigating some separation properties induced by hyper closed functions either on its domain, or range, or on both spaces.

I. Introduction

In our earlier paper [3], we discussed some implications of homomorphism with closed graph in the context of topological groups. In [4] and [5] we have investigated some conditions under which mappings with closed graph on a topological space into another induce some separation properties on its domain and range spaces. Such problems according to the information available with us were initiated by P. E. Long in [2].

The function $f: X \to Y$ has closed graph $G_f = \{(x, f(x)) : x \in X\}$ iff G_f is closed in the product space X x Y. We will denote the interior and the closure of a subset of A by int (A) and cl(A), respectively, and the terminology used is standard. For the sake of convenience, we are giving below some definitions.

We later found in[4] that results obtained in

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[3] get extended to the situation of topological spaces as well and showed that the graph of $f: X \longrightarrow Y$ is closed iff for each $a \in X$, $f(a) = \cap \{cl(f(U)) : U$ is a neighborhood (simply, nbd) of $a\}$. We show in this note that the verification of a similar condition for each compact subset is enough for getting the above equivalence. Further we have tried to investigate some properties of mappings under which such a condition holds for each closed subset of X. We have termed such mappings hyper closed.

Definition 1.1. A mapping $f: X \rightarrow Y$ is said to be nearly continuous [1] iff for every open set V of Y, $f^{-1}(V) \subset int(cl(f^{-1}(V))$.

Definition 1.2. A mapping $f: X \longrightarrow Y$ is called nearly open [1] iff for each open set U of X, $f(U) \subset int(cl(f(U))$.

II. Mappings Having Closed Graph

We begin by quoting a few results proved in [3] and [4] and we find conditions in order that a mapping may have closed graph.

Proposition 2.1. Let $f: G \longrightarrow H$ be a homomorphism with closed graph, G and H being topological groups. Then $e' = \cap \{cl(f(U) : U$ is a nbd of $e\}$ where e and e' are units in G and H, respectively.

Proposition 2.2. Let X and Y be topological spaces and $f: X \longrightarrow Y$ be a mapping. Then the graph of f is closed iff for every $x \in X$, $f(x) = \cap \{cl(f(U)) : U$ is a nbd of $x\}$.

The following proposition generalizes Proposition 2.2.

Proposition 2.3. Amapping $f: X \longrightarrow Y$ has closed graph iff for each compact set A of X, $f(A) = \cap \{cl(f(U_A) : U_A$ is a nbd of $A\}$.

Proof : Let graph of f be closed. Then we have to show that for compact subset A of X, $f(A) = \cap \{cl(f(U_A)) : U_A$ is a nbd of $A\}$. It suffices to show that $\cap \{cl(f((U_A))) : U_A$ is a nbd of $A\} \subset f(A)$. For if not so, there is an element $z \in \cap \{cl(f(U_A)) : U_A$ is a nbd of $A\}$ such that $z \notin f(A)$. By hypothesis, there exist for each $a \in A$, a nbd Ua of a and V_a of z such that $V_a \cap f(U_a) = \phi$. Thus we obtain an open cover $\{U_a : a \in A\}$ of A which has finite subcover. Setting $U_A = \bigcup_{i=1}^{n} \{U_a i : ai \in A\} \supset A$ and $V = \bigcup_{i=1}^{n} \{V_{ai} : ai \in A\}$. We have $V \cap f(U_A) = \phi$, or $z \in cl(f(U_A))$ which is a contradiction and hence $\cap \{cl(f(U_A)) : U_A$ is a nbd of $A\} \subset f(A)$. Thus, $f(A) = \cap \{cl(f(U_A) : U_A$ is a nbd of $A\}$.

The converse part of the proposition follows immediately from the proposition 2.2.

III. Hyper Closed Mappings

Proposition 2.3 leads us to introduce the following definition.

Definition 3.1. A mapping $f: X \rightarrow Y$ is said to be hyper closed if for each closed subset A of X, $f(A) = \cap \{cl(f(U_A) : U_A$ is a nbd of $A\}$.

Hyper cloesed mappings are always closed, but the converse is not true even for homeomorphism.

Example 3.1.1. Let $X = \{a, b, c\}$ be equipped with topology $T = \{X, \phi, \{a\}\}$. Consider the identify mapping $I: X \rightarrow X$. I is even a homeomorphism, but not hyper closed. However, in the positive direction, we obtain the following propositions.

Proposition 3.2. Let X be regular and $f: X \rightarrow Y$ a closed surjection with compact fiber. Then f is hyper closed.

Proof : It suffices to show that for each closed subset A of X, $\cap \{cl(f(U_A : U_A$ is a nbd of $A\} \subset f(A)$. Let $z \notin f(A)$. Then $f^{-1}(z) \cap A = \phi$. X is regular and so for each $x \in f^{-1}(z)$, there exist nbds V_x of X and U_{Ax} of A, respectively, such that $V_x \cap U_{Ax} = \phi$. $f^{-1}(z)$ is compact and so its cover $\{V_x : x \in f^{-1}(z)\}$ has finite subcover with $f^{-1}(z) \subset U^n_{i=1} \{V_{xi} : xi \in f^{-1}(z)\} = V$ (say). Setting $U_A = \cap_{i=1}^{n} \{U_{Axi} : xi \in f^{-1}(z)\}$, we have $U_A \cap V = \phi$, or $f^{-1}(Z) \cap cl(U_A) = \phi$, or $z \notin f(cl(U_A))$, or $z \notin cl(f(U_A))$ for f is closed and hence $\cap \{cl(U_A) : U_A$ is a nbd of $A\} \subset f(A)$ or f is hyper closed.

Similarly, another propositions can be stated.

Proposition 3.3. Let Y be regular and $f : X \longrightarrow Y$ be a closed continuous mapping. Then f is hyper closed.

Proposition 3.4. Let $f : X \longrightarrow Y$ be a continuous mapping from a compact space X into a Hausdorff space Y. Then f is hyper closed.

f is a continuous mapping into Hausdroff space, so it has closed graph and hence the above proposition follows directly from proposition 2.3.

Proposition 3.5. Let $f : X \longrightarrow Y$ be a mapping with closed graph. Then f is hyper closed whenever its graph is compact in X x Y.

Proof : It is given that $G = \{(x, f(x)) : x \in X\}$ is compact. Consider the projection $Px : X \times X \longrightarrow X$ defined by $Px(x, y) = X$. Px is the continuous surjection and hence $Px(G_f) - X$ must be compact, therefore from proposition 2.3, f is hyper closed.

Ⅳ. Some Implications of Hyper Closed Mappings

Proposition 4.1. Let $f : X \longrightarrow Y$. be the given hyper closed nearly continuous injection. Then X is regular.

Proof : Let A be any arbitrary closed set in X and $x \notin A$. Then $f(x) \notin f(A) = \cap\{cl(f(U_A)) : U_A$ is a nbd of A$\}$, or $f(x) \notin cl(f(U_A))$ for at least one nbd U_A of A. There exists a nbd V of f(x) such that $V \cap f(U_A = \phi$, or $f^{-1}(V) \cap U_A = \phi$, or $int(cl(f^{-1}(V))) \cap U_A = \phi$, i.e., f is nearly continuous and hence x and A have disjoint nbds, respectively. So, X is regular.

Corollary 4.1.1. A space X is regular iff the identity mapping $I : X \longrightarrow X$ is hyper closed.

Proof : Let X be regular. Then, by proposition 3.2, I is hyper closed. Conversely, let I be hyper closed. Then by Proposition 4.1, X is regular.

Proposition 4.2. Let $f : X \longrightarrow Y$ be a hyper closed continuous nearly open surjection. Then Y is regular.

Proof : Let A be any closed set in Y such that $y \notin A$ or $y \notin f(f^{-1}(A))$ for f is surjective. Since f is continuous and hyper closed, there exists at least one nbd U of $f^{-1}(A)$ such that $y \notin cl(f(U))$ *or there exists a nbd V of y such tat* $V \cap f(U) = \phi$ since f is nearly open, so $V \cap f(U) = \phi$ implies $V \cap int(cl(f(U))) = \phi$, that is, y and A have disjoint nbds and hence Y is regular.

Combining above results, we have the following.

Proposition 4.3. Let $f : X \rightarrow Y$ be a hyper closed homeomorphism from X onto Y. Then both X and Y are regular spaces.

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