울산대학교 자연과학연구논문집 제13권 1호 pp. 49~54, 울산대학교.
Journal of Natural Science Vol. 13, No. 1, pp. 49~54, 2003
University of Ulsan.

A BOUND OF DISCONTINUOUS GALERKIN SOLUTIONS FOR THE CAHN-HILLIARD EQUATION

S.M. CHOO

Department of Mathematics, University of Ulsan
Ulsan 680-749, Korea
e-mail: smchoo@mail.ulsan.ac.kr

ABSTRACT. Cahn-Hilliard equation is modeled to describe the phase separation in systems. Using a discontinuous Galerkin method, we will control accuracy of approximate solutions. In this paper, boundedness of discontinuous Galerkin solution is discussed. A posteriori error estimation of the Cahn-Hilliard equation, which is based on the results of this paper, will be studied later.

1. Introduction.

Consider the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} - \delta\frac{\partial^2 u_t}{\partial x^2} + \alpha\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 \phi(u)}{\partial x^2}, \quad x \in \Omega, \quad 0 < t, \tag{1.1a}$$

with an initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega = (0,1), \tag{1.1b}$$

and boundary conditions

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \partial\Omega, \quad 0 < t. \tag{1.1c}$$

Here The function $\phi(u) = \gamma u^3 - \beta^2 u$ is an intrinsic chemical potential and δ and α are positive coefficients of viscosity and gradient energy, respectively. And $u(x,t)$ is the concentration of one of two components of the system.

The equation (1.1) with $\delta > 0$ arises as a phenomenological continuum model for phase separation in glass and polymer systems where intermolecular friction forces may be expected to be of importance. See Novick-Cohen[14] for a derivation of the model and

---

1991 *Mathematics Subject Classification.* Primary 65M12, 65M15, 65M60.

Key words and phrases. Cahn-Hilliard equation, a posteriori error, dual problem, discontinous Galerkin method.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Novick-Cohen and Pego [15] for more physical motivation. The viscous Cahn-Hilliard equation, which is viewed as a singular limit of the phase field model of phase transition, has been studied by Bai, Elliott, Gardiner, Spence, and Stuart [2]. They have studied the similarities and differences between the Cahn-Hilliard equation($\delta = 0$) and Allen-Cahn equation by using the viscous Cahn-Hilliard equation. Metastable pattern for the viscous Cahn-Hilliard equation has been studied by Reyna and Ward[16]. Using explicit energy calculations, Grinfeld and Novick-Cohen[13] have established a Morse decomposition of the stationary solutions of the viscous Cahn-Hilliard equation. Existence theory of the solution of (1.1) has been shown in Elliott and Stuart[9]. Choo and Chung[3] have investigated the exponential decay of the classical solutions of (1.1) and compared decay speeds of the viscous Cahn-Hilliard equation with that of the Cahn-Hilliard equation analytically.

Compared to numerical studies for the Cahn-Hilliard equation by Elliot and French[6]-[7], Elliot, French and Milner[8] with finite element methods and Furihata, Onda, and Mori[11], Sun[18], Choo and Chung[4], Choo, Chung and Kim [5] with finite difference methods, there is no numerical study for the viscous Cahn-Hilliard equation.

Generally, the a priori error bounds depend on the exact solution u. But if the exact solution u of (1.1) has very steep gradients and curvatures, then the a priori error may be extremely large even though mesh size is very small. Thus it is natural to refine the grid size in order to increase the accuracy. However, since the nature of exact solution u is unknown, it is not clear how to locally refine the finite element mesh. For the control of mesh refinement, a posteriori error estimates expressed in terms of only the data of the problem and of the computed solution are studied using the method of residual and the method of dual problem. We refer to Ainsworth and Oden[1], Eriksson and Johnson[10], Grasselli, Perotto and Saleri[12] and references therein.

In order to derive a posteriori error estimates for the problem (1.1), we consider discontinuous Galerkin(DG) method, which is to find approximating function discontinuous in time and continuous in space. The process of the a posteriori error estimates is sketched as follows.

(1) Represent the error in terms of the solution of a dual continuous problem.
(2) Determine certain constants in the a posteriori error bounds using Galerkin orthogonality with local interpolation estimates and stability estimate for the dual continuous problem.

The main purpose of this paper is to obtain a bound of the discontinous Galerkin solution of Cahn-Hilliard equation. The layout of this paper is as follows. Basic notations and some preliminary results are introduced in Section 2. In Section 3, discontinuous Galerkin approximations are introduced and the stability of discontinuous solutions are established. In later research, we will study a posteriori error estimation of the Cahn-Hilliard equation, using a dual problem.

2. Notations and preliminaries.

The standard notations for Sobolev spaces and norms will be used. In particular, L^2 denotes $L^2(\Omega)$ space with $(\cdot,\cdot)$ as an inner product and $\|\cdot\|$ as an induced norm. For a nonnegative integer k, H^k stands for Sobolev Hilbert space $H^k(\Omega)$ with norm $\|\cdot\|_k$. Further, let $H_0^1 = \{v \in H^1 : v = 0 \text{ on } \partial\Omega\}$ and $H_0^2 = \{v \in H^2 : v = 0, v_x = 0 \text{ on } \partial\Omega\}$.

Taking inner product of (1.1) with $v \in H_0^2$ and applying boundary condition (1.1c), we obtain the weak formulation

$$(u_t, v) - \delta(u_{xxt}, v_{xx}) + \alpha(u_{xx}, v_{xx}) = (\phi(u)_{xx}, v), \quad v \in H_0^2 \tag{2.1}$$

with $u(0) = u^0$.

Existence and uniqueness of the solution for (2.1) is shown in Choo and Chung[3].

Lemma 2.1. *Given $u_0 \in H_0^2$, there exists a unique global weak solution u of (2.1) such that for a constant C*

$$\|u\|_{L^\infty(H^2)} \leq C\|u_0\|_2,$$

where $\|u\|_{L^\infty(H^2)} = sup_{t\in[0,T]}\|u(\cdot,t)\|_2$.

Let V_h be a finite dimensional subspace of $H_0^1 \cap H^2$ with the following approximation property: there exists a constant C independent of spatial mesh size h such that for $u \in H_0^2 \cap H_4$,

$$\inf_{\chi \in V_h} \|u - \chi\|_j \leq Ch^{r-j}\|u\|_r \quad j = 0, 1, 2, \quad 2 \leq r \leq 4. \tag{2.2}$$

Introduce a bilinear form

$$A(v, w) = (v, w) + \delta(v_x, w_x), \quad v, w \in H_0^2,$$

and an energy norm

$$\|v\|_A = A(v, v)^{\frac{1}{2}}, \quad v \in H_0^2.$$

Then we obtain the following lemma using the Cauchy inequality.

Lemma 2.2. *For any $v, w \in H_0^2$, the inequality*

$$A(v, w) \leq \|v\|_A\|w\|_A$$

holds.

Let $\tilde{u} \in V_h$ be an auxiliary projection of u, defined by

$$A(u - \tilde{u}, v) = 0, \quad v \in V_h. \tag{2.3}$$

Then the energy norm of projection is dominated by that of original function.

Lemma 2.3. *For any projection $\tilde{u}$ of u, the inequality*

$$\|\tilde{u}\|_A \leq \|u\|_A$$

holds.

Using (2.2)–(2.3), we obtain the following estimates.

Lemma 2.4. *There exists a constant C such that*

$$\|u - \tilde{u}\| \leq Ch^2\|u_{xxxx}\|$$

and

$$\|(u - \tilde{u})_t\| \leq Ch^2\|u_{xxxxt}\|.$$

3. Discontinuous Galerkin method.

Define a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $I = (0, T]$ into subintervals $I_n = (t_{n-1}, t_n]$ of length $k_n = t_n - t_{n-1}$. For a given function $v(t)$, let

$$v^{n,+} = \lim_{s \to 0+} v(t_n + s), \quad v^{n,-} = \lim_{s \to 0-} v(t_n + s),$$

and $[v^n] = v^{n,+} - v^{n,-}$ denote the jump of v at time t_n.

We introduce a spatial discretization on each slab $S_n = \Omega \times I_n$ based on conforming finite elements. For each $n = 1, 2, \cdots, N$, let $\{x_i^n\}$ be a partition of Ω into intervals (x_{i-1}^n, x_i^n) with $h_n^i = x_i^n - x_{i-1}^n$. We also introduce the global mesh function $h = h(x,t)$ defined by $h(x,t) = h_n$ for $x \in \Omega, t \in I_n$ and $k = k(t)$ by $k(t) = k_n$ for $t \in I_n$. Finally, we require the quasi-uniformity of the meshes.

Let $J_n : L^2(I_n) \to \mathbb{P}_1(I_n)$ be the L^2-projection onto the set $\mathbb{P}_1(I_n)$ of linear functions on I_n. Then the following lemma on projection errors holds.

Lemma 3.1. *For $1 \le p \le \infty$, there is a constant γ such that*

$$\|v - J_n v\|_{L^p(I_n)} \le \gamma \min\{k_n \|v_t\|_{L^p(I_n)}, k_n^2 \|v_{tt}\|_{L^p(I_n)}\}$$

and

$$\|(v - J_n v)^{n-1,+}\| \le \gamma \min\{\|v_t\|_{L^1(I_n)}, k_n \|v_{tt}\|_{L^1(I_n)}\}.$$

In order to get our discretization, we will use the following notations

$$W_n = \{w : S_n \to \mathbb{R} | w(x,t) = \phi_{n0} + t\phi_{n1}, \quad \phi_{n0}, \phi_{n1} \in V_{h_n}, (x,t) \in S_n\}$$

and

$$W = \{w : \Omega \times I \to \mathbb{R} | \quad w|_{I_n} \in W_n, n = 1, 2, \cdots, N\}.$$

Integrating (2.1) by parts in a fixed interval $[0, t_N]$, we obtain

(3.1)
$$- \int_0^{t_N} A(u, w_t)dt + A(u(t_N), w(t_N)) - A(u^0, w(0)) + \alpha \int_0^{t_N} (u_{xx}, w_{xx})dt$$
$$- \int_0^{t_N} (\phi(u)_{xx}, w)dt = 0, \quad \forall w \in W.$$

Note that

$$\begin{aligned} - \int_0^{t_N} A(U, w_t)dt &= - \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} A(U, w_t)dt \\ &= \sum_{n=1}^{N} \int_{I_n} A(U_t, w)dt + \sum_{n=2}^{N} A(U^{n-1,+} - U^{n-1,-}, w^{n-1}) \\ &\quad - A(U^{N,-}, w^N) + A(U^{0,+}, w^0), \end{aligned}$$

where $w^n = w(t_n)$. It follows from (3.1) that

(3.2)
$$\sum_{n=1}^{N}\int_{I_n} A(U_t,w)dt+\sum_{n=1}^{N} A([U^{n-1}],w^{n-1})+\sum_{n=1}^{N}\int_{I_n}(U_{xx},w_{xx})dt-\sum_{n=1}^{N}\int_{I_n}(\phi(U)_{xx},w)dt=0,$$
with $U^{0,-}=u^0$.

Since a function w in W is not required to be continuous at t_n, we may choose its values on the different time intervals independently. By choosing w to be vanished outside I_n, we can introduce the space-time discretization of the Cahn-Hilliard equation to find $U \in W$ such that

(3.3)
$$\int_{I_n} A(U_t,w)dt+A([U^{n-1}],w^{n-1,+})+\alpha\int_{I_n}(U_{xx},w_{xx})dt-\int_{I_n}(f(U)_x,w)dt=0, \quad \forall w \in W_n,$$
with
$$U^0 = u^0.$$

Remark 3.1. The problem to find $U \in W$ satisfying (3.3) is called a cGdG-method, since U is contionuous in space and discontinuous in time.

Since the bilinear form is positive definite, existence and uniqueness of a solution for (3.3) can be shown in a standard manner. We here show that the solution U of (3.3) is stable.

Theorem 3.1. *Let $\alpha \geq \frac{\beta^2}{\pi^2}$ and U be the solution of (3.3). Then for $n = 1, 2, \cdots, N$*
$$\|U^{n,-}\|_A \leq \|u^0\|_A.$$

Proof. If we take $w = U$ in (3.3), then

$$\text{(3.4)}\quad \frac{1}{2}\int_{I_n}\frac{d}{dt}\|U\|_A^2 dt+\|U^{n-1,+}\|_A^2+\int_{I_n}\alpha\|U_{xx}\|^2-(\phi(U)_{xx},U)dt = A(U^{n-1,-},U^{n-1,+}).$$
Note that
$$-(\phi(U)_{xx},U) \geq \beta^2(U_{xx},U) = -\beta^2\|U_x\|^2 \geq -\frac{\beta^2}{\pi^2}\|U_{xx}\|^2.$$
Since $\int_{I_n}\frac{d}{dt}\|U\|_A^2 dt = \|U^{n,-}\|_A^2 - \|U^{n-1,+}\|_A^2$ and $\alpha \geq \frac{\beta^2}{\pi^2}$, it follows from (3.4) that
$$\|U^{n,-}\|_A^2 + \|U^{n-1,+}\|_A^2 \leq 2\|U^{n-1,-}\|_A\|U^{n-1,+}\|_A \leq \|U^{n-1,-}\|_A^2 + \|U^{n-1,+}\|_A^2.$$
Hence we obtain
$$\|U^{n,-}\|_A^2 \leq \|U^{n-1,-}\|_A^2,$$
which completes the proof. □

Remark 3.2. From the above theorem, we may also show that the inequality holds
$$\|U^{n,+}\|_A^2 \leq 6\|u^0\|_A^2.$$

Remark 3.3. Using the Sobolev embedding theorem, we can show that
$$\|U\|_{L^\infty(L^\infty)} \leq \max_{1\leq n\leq N}\max\{\|U^{n,-}\|_A, \|U^{n-1,+}\|_A\} \leq C\|u^0\|_2.$$
Hence the inequality $\|f(U)\|_{L^\infty(L^\infty)} \leq C\|u^0\|_2$ follows.

REFERENCES

1. M. Ainsworth and J.T. Oden, *A posteriori error estimation in finite element analysis*, Comput. Meth. Appl. Mech. Eng. **142**, 1–88(1997).
2. F. Bai, C. M. Elliott, A. Gardiner, A. Spence and A. M. Stuart, The viscous Cahn-Hilliard equation. Part I: computations, *Nonlinearity* **8**, 131–160(1995).
3. S. M. Choo and S. K. Chung, Asymptotic behavior of the viscous Cahn-Hilliard equation with the Cahn-Hilliard equation, submitted.
4. S. M. Choo and S. K. Chung, Conservative nonlinear difference scheme for the Cahn-Hilliard equation, *Comp. Math. Appl.* **36**, 31–39 (1998).
5. S. M. Choo, S. K. Chung and K. I. Kim, Conservative nonlinear difference scheme for the Cahn-Hilliard equation:II, *Comp. Math. Appl.* **39**, 229-243 (2000).
6. C. M. Elliott and D. A. French, Numerical studies of the Cahn-Hilliard equation for phase separation, *IMA J. Appl. Math.* **38**, 97–128(1987).
7. C. M. Elliott and D. A. French, A nonconforming finite-element method for the two-dimensional Cahn-Hilliard equation, *SIAM J. Numer. Anal.* **26**, 884–903(1989).
8. C. M. Elliott, D. A. French and F. A. Milner, A second order splitting method for the Cahn-Hilliard equation, *Numer. Math.* **54**, 575–590(1989).
9. C. M. Elliott and A. M. Stuart, Viscous Cahn-Hilliard equation II. analysis, *J. Diff. Eq.* **128**, 387–414(1996).
10. K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problem IV: Nonlinear prob lems*, SIAM J. Numer. Anal. **32**, 1729–1749(1995).
11. D. Furihata, T. Onda and M. Mori, A finite difference scheme for the Cahn-Hilliard equation based on a Lyapunov functional, *GAKUTO Int. Series, Math. Sci. Appl.* **2**, 347–358(1993).
12. M. Grasselli, S. Perotto and F. Saleri, *Space-time finite element approximation of Boussinesq equations*, East-West J. Numer. Math. **7**, 283–306(1999).
13. M. Grinfeld and A. Novick-Cohen, The viscous Cahn-Hilliard equation: Morse decomposition and structure of the global attractor, *Trans. Amer, Math. Sci.* **351**, 2375–2406(1999).
14. A. Novick-Cohen, On the viscous Cahn-Hilliard equation, In *Material Instabilities in Continuum and Related Mathematical Problems* (edited by J.M. Ball), Oxford Univ.Press, Oxford(1988).
15. A. Novick-Cohen and R. L. Pego, Stable patterns in a viscous diffusion equation, *Trans. Amer. Math. Soc.* **324**, 331–351(1991).
16. L. G. Reyna and M. J. Ward, Metastable internal layer dynamics for the viscous Cahn-Hilliard equation, *Methods Appl. Anal.* **2**, 285–306 (1995).
17. Z. Z. Sun, A second-order accurate linearized difference scheme for the two-dimensional Cahn-Hilliard equation, *Math. Comp.* **64**, 1463–1471(1995).