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A note on a definition of Riemann integrable

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〈**Abstract**〉

Cauchy's definition for the Riemann integral is non-sense, for the Cauchy's sum $\sum_{i=1}^{n} f(\xi_i)\Delta xi$, where $\xi_i \in [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$, of with respect to variable n, and more over it is not a net or a filter, hence we are not able to consider the $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$. In this note we try to define the Riemann integral in terms of limit process more strictly.

Riemann 적분 정의에 관하여

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〈요　약〉

Cauchy의 Riemann 적분에 관한 정의는 오류가 있다. 왜냐하면 Cauchy의 합, 즉 $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$는 변수 n에 관하여 수열도 함수도 아니며, net도 filter도 아니므로 Cauchy의 합의 극한을 생각할 수 없기 때문이다. 이러한 오류를 시정하기 위하여 filter의 극한에 대한 이론으로서 Riemann 적분을 정의한다.

I. Introduction

It is well known that Cauchy defined the integral(in the sense of Riemann) as follows.

Let $[a,b]$ be a given interval. By a subdivision P of $[a,b]$, we mean a finite set $\{x_0, x_1, x_2, \cdots, x_n\}$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We write $\Delta x_i = x_i - x_{i-1} (i = 1, \cdots, n)$.

If limit of Cauchy's sum $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$, where $\xi_i \in [x_{i-1}, x_i]$, then f is said to be integrable on $[a,b]$ and denote it by $\lim_{\substack{n\to\infty \\ \delta\to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx$, where $\delta = \max\{\Delta x_i\}$.

But this definition is non-sense because $\lim_{\substack{n\to\infty \\ \delta\to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ has no means. In the theory of analysis we can consider the sequences, functions, nets or filters. For a fixed n there are various kinds of subdivision P of $[a,b]$. Its cardinality $\aleph\{P\}$ is precisely $\aleph_1$. And for each subdivision $P = \{a = x_0, x_1, \cdots, x_n = b\}$ method of choosing $\{\xi_1, \xi_2, \cdots \xi_n\}$ is not unique. It is clear that $\aleph\{(\xi_1, \xi_2, \cdots \xi_n) | [x_{i-1}, x_i], i = 1, 2, \cdots, n\}$ is also $\aleph_1$.

Therefore each fixed n, the cardinality of the set of number $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ is $\aleph_1 \times \aleph_1 = \aleph_1$. So the Cauchy sum $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ is not a sequence or function of with respect to variable n. Moreover it is not a net or a filter.

In the above sense we are not able to consider the $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$.

In this note we try to define the Riemann integral in terms of limiting process more strictly.

Ⅱ. Filters

Definition 1. A filter on a set X is a family $\mathscr{F}$ of subsets of X which has following properties;

i) $\mathscr{F} \neq \phi$, $\phi \notin \mathscr{F}$

ii) For each pair (A, B) of subsets A, B of X, $A \cap B \in \mathscr{F}$ $A, B \in \mathscr{F}$

Properies i) and ii) in the above definition is equivalent to the following properties;

F_1) Every subset of X which contains a member of $\mathscr{F}$ belongs to $\mathscr{F}$.

F_2) Every finite intersection of members of $\mathscr{F}$ belongs ot $\mathscr{F}$.

F_3) The empty set is not in $\mathscr{F}$.

In a topological space X, the set of all neighbourhoods of an arbitrary non-empty subset A of X (and in particular the set of all neighbourhoods of a point of X) is a filter, called the neighbourhood filter of A.

Definition 2. Given two filters $\mathscr{F}$, $\mathscr{F}'$ on the same set X, $\mathscr{F}'$ is said to be finer than $\mathscr{F}$, or $\mathscr{F}$ is coarser than $\mathscr{F}'$, if $\mathscr{F} \subset \mathscr{F}'$. If also $\mathscr{F} \neq \mathscr{F}'$, than $\mathscr{F}'$ is said to be strictly finer than $\mathscr{F}$, or $\mathscr{F}$ strictly coarser than $\mathscr{F}'$.

Two filters are said to be comparable if one is finer than the other.

The set of all filters on X is ordered by the relation "$\mathscr{F}$ is coarser than $\mathscr{F}'$"; this relation is induced by the inclusion relation in ($\mathscr{B}$ ($\mathscr{B}$ (X).

Let $(\mathscr{F}_i)_{i \in I}$ be any non-empty family of filters on a set X (which must therefore be non-empty); then the set $\mathscr{F} = \bigcap_{i \in I} \mathscr{F}_i$ satisfies anxioms(F_1), (F_2) and(F_3) and is therefore a filter; $\mathscr{F}$ is called the intersection of a family of filters $(\mathscr{F}_i)_{i \in I}$Land is obviously the greatest lower bound of the set of the $\mathscr{F}_i$ in the ordered set of all filters on X.

Given a family $\mathscr{G}$ of subsets is a set of X, let us consider whether there are any filters on X which contain $\mathscr{G}$. If such a filter exists then by (F_2) it contains also the family $\mathscr{G}$ of finite intersections of members of $\mathscr{G}$ (including X, which is the intersection of the empty sub-family of $\mathscr{G}$); hence a necessary condition for such a filter to exist is that the empty subset of X is not is $\mathscr{G}$. This condition is also sufficient, for by (F_1) and filter which contains $\mathscr{G}'$ also contains the family $\mathscr{G}''$ of subsets of X which contain a member of $\mathscr{G}'$. Now $\mathscr{G}''$ cleary satisfies (F_1); it satisfics (F_2) by reason the definition of $\mathscr{G}'$; and finally it satisfies (F_3) because the empty subset of X doesnot belong to $\mathscr{G}'$. Hence $\mathscr{G}''$ is the coarsest filter which contains $\mathscr{G}$, and we have proved:

Lemma 1. A necessary and sufficient condition that there should exist a filter on X containing a family $\mathscr{G}$ of subset of X is that no finite sub-family of $\mathscr{G}$ has an empty intersection. The filter $\mathscr{G}''$ defined above is said to be generated by $\mathscr{G}$, and $\mathscr{G}$ is said to be a subbase of $\mathscr{G}''$.

If $\mathscr{G}$ is a subbase of a filter $\mathscr{F}$ on X, then $\mathscr{F}$ is not in general the family of substs of X which contain a set of $\mathscr{G}$; for $\mathscr{G}$ to have this property it is necessary and sufficient that every finite intersection of sets of $\mathscr{G}$ should contain a set of $\mathscr{G}$. Hence we have following Lemma:

Lemma 2. Let $\mathscr{B}$ be a family of subsets of a set X, then the set of subsets of X which contain a set of $\mathscr{B}$ is a filter if and only if $\mathscr{B}$ has the following two properties;

(B_1) The intersection of two sets of $\mathscr{B}$ contains a set of $\mathscr{B}$.

(B_2) $\mathscr{B}$ is not empty, and the empty subset of X is not in $\mathscr{B}$.

Definition 3. A family $\mathscr{B}$ of subsets of a set X which satisfies axiom(B_1) and (B_2) is said to be a base of the filter it generates; two filter bases are said to be equivalent if they generate the same filter.

If $\mathscr{G}$ is subbase of a filter $\mathscr{F}$, then the family $\mathscr{G}'$ of intersection of finite sub-family of $\mathscr{G}$ is a base of $\mathscr{F}$.

Lemma 3. A subfamily $\mathscr{B}$ of a filter $\mathscr{F}$ on X is a base of $\mathscr{F}$ if and only if every member of $\mathscr{F}$ contains a member of $\mathscr{B}$. If $\mathscr{B}$ is a base of $\mathscr{F}$, then clearly every member of $\mathscr{F}$ contains a member of $\mathscr{B}$: conversely, if every member of $\mathscr{F}$ contains a member of $\mathscr{B}$, then the family of subsets of X containing a member of $\mathscr{B}$ coincides with $\mathscr{F}$ by reason of (F_1).

Lemma 4. On a set X, a filter $\mathscr{F}'$ with base $\mathscr{B}'$ finer than a filter $\mathscr{F}$ with base $\mathscr{B}$ if and only if every member of $\mathscr{B}$ contains a member of $\mathscr{B}'$.

This is an immediate consequence of difinition 2 and 3.

Corollary. Two filter bases $\mathscr{B}$, $\mathscr{B}'$ on a set X are equivalent if and only if every member of $\mathscr{B}$ contains a member of $\mathscr{B}'$ and every member of $\mathscr{B}'$ contains a member of $\mathscr{B}$.

Definition 4. An ultrafilter on a set X is a filter $\mathscr{F}$ such that there is no filter on X which is strictly finer than $\mathscr{F}$ (in other words, a maximal element in the ordered set of all filters on X).

Since the ordered set of all filters on X is inductive, Zorns' lemma shows that:

Lemma 5. If $\mathscr{F}$ is any filter on a set X, there is an ultrafilter finer than $\mathscr{F}$.

Lemma 6. Let $\mathscr{F}$ be an ultrafilter on a set X. If A and B are two subset of X such that $A\cup B\in\mathscr{F}$, then either $A\in\mathscr{F}$ or $B\in\mathscr{F}$.

[Proof] If the lemma is false, there exists A and B of X such that $A\notin\mathscr{F}$ and $B\notin\mathscr{F}$ and $A\cup B\in\mathscr{F}$. Let $\mathscr{G}$ be the family of subsets M of X such that $A\cup M\in\mathscr{F}$. It is straightforward to check that $\mathscr{G}$ is a filter on X, and $\mathscr{G}$ is strictly finer than $\mathscr{F}$, since $B\in\mathscr{G}$ but this contradicts the hypothesis that $\mathscr{F}$ is an ultrafilter

Corollary. If the union of a finite sequence $(Ai)_{1\leq i\leq n}$ of subsets of X belongs to an ultrafilter $\mathscr{F}$, then at least one of the Ai belongs to $\mathscr{F}$.

Lemma 7. Let $\mathscr{G}$ be a subbase of a filter on a set X. If for each subset Y of X we have either $Y\in\mathscr{G}$ or $CY\in\mathscr{G}$, then $\mathscr{G}$ is an ultrafilter on X.

[Proof] Let $\mathscr{F}$ be a filter containing $\mathscr{G}$(there is one, by hypothesis); then $\mathscr{F}$ coincides with $\mathscr{G}$; for it $Y\in\mathscr{F}$; then $CY\notin\mathscr{F}$; Hence $CY\notin\mathscr{G}$ and there $Y\in\mathscr{G}$.

Lemma 8. Every filter $\mathscr{F}$ on a set X is the intersection of the ultrafilters finer than $\mathscr{F}$.

[Proof] Clearly this intersection contains $\mathscr{F}$. Convrsely, let A be a subset of X which doesnot belong to $\mathscr{F}$, and let A' denote CA; A contains no set of $\mathscr{F}$; hence every $M\in\mathscr{F}$ meats A' and therefore there is a filter $\mathscr{F}'$ which is finer than $\mathscr{F}$ and contains A'. If $\mathscr{U}$ is an ultrafilter finer than $\mathscr{F}$ (Lemma5), it follows that $A\in\mathscr{U}$. This completes the proof.

Lemma 9. Let $\mathscr{F}$ be a filter on a set X and A a subset of X. Then the trace $\mathscr{F}_A=\{A\cap F|F\in\mathscr{F}\}$ of $\mathscr{F}$ on A is a filter if and only if each member of $\mathscr{F}$ meets A.

Since $(M\cap N)\cap A=(M\cap A)\cap(N\cap A)$ we see that $\mathscr{F}_A$ satisfies (F_2); again, if $M\cap A\subset P\subset A$ then $P=(M\cup P)\cap A$, Whence $\mathscr{F}_A$ satisfies (F_1). Hence $\mathscr{F}_A$ is a filter if and only if it satisfies(F_3), i.e if and only if each member of $\mathscr{F}$ meets A. In particular, if $A\in\mathscr{F}$ then $\mathscr{F}_A$ is a filter on A, by (F_2) and (F_3).

Definition 5. Let A be a subset of a set X and $\mathscr{F}$ a filter on X.

If the trace of $\mathscr{F}$ on A is a filter on A, this filter is said to be induced by $\mathscr{F}$ on A.

If a filter $\mathscr{F}$ on X induces a filter on $A\subset X$, then the trace on A of a base of $\mathscr{F}$ is base of $\mathscr{F}_A$, by reason of Lemma3.

Lemma 10. An ultrafilter $\mathscr{U}$ on a set X induces a subset A of X if and only if $A\in\mathscr{U}$; and if this condition is satisfied then $\mathscr{U}_A$ is an ultrafilter on A.

This is an immediate consequence of proposition 6 and 7.

Definition 6. Let X be a topological space and $\mathscr{F}$ a fillter on X. A point $x \in X$ is said to be a limit point (or simply a limit) of $\mathscr{F}$, if $\mathscr{F}$ is finer than the neighborhood filter $\mathscr{B}(x)$ of x; $\mathscr{F}$ is also said to converge (or to be convergent) to x. The point x is said to be a limit of a filter base $\mathscr{B}$ on X, and $\mathscr{B}$ is said to converge to x, if the filter whose base is $\mathscr{B}$ converges to x.

This definition, together with proposition 4, gives the following criterion:

Lemma 11. A filter base $\mathscr{B}$ on a topological space X converges to x if and only if every set of a fundamental system of neighbourhoods of x contains a member of $\mathscr{B}$.

If a filter $\mathscr{F}$ converges to x, then every filter than $\mathscr{F}$ also converges to x, by reason of Definition 6. Like wise, if the topology of X is replaced by a coarser topology, the neighbourhood filter of x is replaced by a coarser filter and therefore $\mathscr{F}$ still converges to x in this new topology. Let ϕ be a family of filter on X, all of which converge to the same point x; the neighbour hood filter $\mathscr{B}(x)$ is coarser than all the filters of ϕ, hence also coarser than the intersection $\mathscr{F}$ of these filters; In other words, $\mathscr{F}$ also converges to x.

Lemma 12. A filter $\mathscr{F}$ on a topological space X converges to point x if and only if every ultrafilter which is finer than $\mathscr{F}$ converge to x.

This is an immediate consequence of Lemma 8.

III. Riemann integral

Suppose that $f:\mathscr{R}\to\mathscr{R}$ is a bounded real valued function defined on $[a,b]$. Corresponding to each subdivision $P=\{a=x_0, x_1, x_2, \cdots, x_n=b\}$ of $[a,b]$, we put $Mi=\sup\{f(x)\mid x\in[x_{i-1}, x_i]\}$

$mi=\inf\{f(x)\mid x\in[x_{i-1}, x_i]\}$

$U(P,\mathrm{f})=\sum_{i=1}^{n} M_i \Delta x_i$: upper sum

$L(P,f)=\sum_{i=1}^{n} m_i \Delta x_i$: lower sum, where $\Delta x_i=x_i-x_{i-1}$,

and finally

(1) $\overline{\int_a^b} f(x)\,dx=\inf U(P,f)$, (upper Riemann integrals of f)

(2) $\underline{\int_a^b} f(x)dx=\sup L(P,f)$, (lower Riemann integrals of f).

If $\overline{\int_a^b} fdx=\underline{\int_a^b} fdx$, we say that f is Riemann integrable on $[a,b]$, and the common value (1) and (2) is denoted by $\int_a^b fdx$.

Let $f:[a,b]\to\mathscr{R}$ be given, and let $\alpha=\{P\mid P=\{a=x_0, x_i, \cdots, x_n=b\}$ is a subdivision of a,b. Then is directed by the inclusion order.

Theorem 1. For each subdivision $P=\{a=x_0, x_1, \cdots x_n=b\}$, let

$R_P=\{\sum_{i=1}^{n} f(\xi_i)\Delta x_i \mid \xi_i\in[x_{i-1}, x_i]\ i=1,2,\cdots,n\}$, and define

$Ap=\bigcup\{R_Q\mid P\subset Q,\ Q\in\alpha\}$, then $\mathscr{F}=\{B\mid A_P\subset B$ for some $P\in\alpha\}$ is a filter.

[Proof] It is enough to show that $\mathscr{L}=\{A_P\mid p\in\alpha\}$ is a filterbase.

It is clear the $\mathscr{L}\neq\phi$ and $\phi\notin\mathscr{L}$, now let A_P, $A_Q\in\mathscr{L}$, then $P\cup Q=S\in\alpha$ and $P\subset S$, $R\subset S$ and $A_S\subset A_P$ and $A_S\subset A_Q$. Hence $A_P\cap A_Q\supset A_S$. This means that $\mathscr{L}$ is a base for the filter $\mathscr{F}$.

Theorem 2. A bounded real valued function $f:[a, b]\to\mathscr{R}$ is Riemann integrable on $[a,b]$ if and only if the filter $\mathscr{F}=\{B\mid A_P\subset B$ for some $P\in\alpha\}$ converges.

[Proof] ($\Rightarrow$) Let $\underline{\int_a^b} f(x)dx=\overline{\int_a^b} f(x)dx$, then

$$\sup_{p\in\alpha}\left\{\sum_{i=1}^{n} m_i\,\Delta x_i\right\}=\inf_{p\in\alpha}\left\{\sum_{\tau=1}^{n} M_i\,\Delta x_i\right\}=\alpha.$$

Now let $(\alpha-\varepsilon,\ \alpha+\varepsilon)$ be any basic ε neighbourhood of α there are subdivision $P=\{a=x_0, x_1, \cdots, x_n=b\}$ and $Q=\{a=x_0', x_1', \cdots, x_m'=b\}$ of $[a,b]$

such that

$$\alpha-\varepsilon<\sum_{i=1}^{n} m_i\,\Delta x_i\leq\alpha,\quad \alpha\leq\sum_{j=1}^{m} M_j'\,\Delta x_j'<\alpha+\varepsilon.$$ Since

$mi=\inf\{f(x)\mid x\in[x_{i-1}, x_i]\}$,

$M_j'=\sup\{f(x)\mid x\in x_{j-1}', x_j'\}$, for all choices$(\xi_1,\cdots\xi_n)$ where $\xi_i\in[x_{i-1},x_i]$, and $(\xi_i',\cdots,\xi_m'\}$, where $\xi_j'\in[x_{j-1}', x_j']$.

We have

$$\alpha-\varepsilon<\sum_{i=1}^{n} m_i\,\Delta x_i\leq\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

and

$$\sum_{i=1}^{m} f(\xi_j')\ x_j'\leq\sum_{j=1}^{m} M_j'\,\Delta x_j'<\alpha+\varepsilon.$$

Let $S=P\cup Q$, then $S\in\mathscr{A}$, and let $S=\{a=y_0, y_1, \cdots, y_l=b\}$then $P\subset S$, $Q\subset S$, Hence

$$\begin{aligned}\alpha-\varepsilon &<\sum_{i=1}^{n} m_i\,\Delta x_i\leq\sum_{k=1}^{l} m_k''\,\Delta y_k\\ &\leq\sum_{k=1}^{l} f(\xi_k'')\,\Delta y_k\leq\sum_{k=1}^{l} M_k''\,\Delta y_k\\ &\leq\sum_{j=1}^{m} M_j'\,\Delta x_j'<\alpha+\varepsilon,\end{aligned}$$

where $m_k''=\inf\{f(x)\mid x\in[y_{k-1}, y_k]\}$

$M_k''=\sup\{f(x)\mid x\in[y_{k-1}, y_k]\}$, and $\xi_k''\in[y_{k-1}, y_k]$.

Therefore we have $A_S=\{R_T\mid S\subset T\}\subset(\alpha-\varepsilon,\alpha+\varepsilon)$, $A_S\in\mathscr{S}$.

That is, $(\alpha-\varepsilon,\alpha+\varepsilon)\in\mathscr{F}$ or $\mathscr{F}$ converges to α.

($\Leftarrow$) Suppose that $\mathscr{F}$ converges to $\alpha\in\mathscr{R}$.

Then for each $\frac{\varepsilon}{2}-nbd$

$$N_{\frac{\varepsilon}{2}}(\alpha)=\left(\alpha-\frac{\varepsilon}{2},\quad \alpha+\frac{\varepsilon}{2}\right) \text{ of } \alpha,$$

$$N_{\frac{\varepsilon}{2}}(\alpha)\in\mathscr{F}.$$

Therefore there is a member $P\in\mathscr{A}$ such that

$A_P\subset N_{\frac{\varepsilon}{2}}(\alpha)$, This means that

$$R_P=\left\{\sum_{i=1}^{n} f(\xi_i)\Delta x_i\mid \xi_i\in[x_{i-1}, x_i]\right\}\subset N_\varepsilon(\alpha),$$

$$P=\{a=x_0, x_1,\cdots, x_n=b\}.$$

Consequently $\sum_{i=1}^{n} m_i\,\Delta x_i$ and $\sum_{i=1}^{n} M_i\,\Delta x_i$ belong to $(\alpha-\varepsilon,\ \alpha+\varepsilon)$.

That is,

$$\sum_{i=1}^{n} m_i\,\Delta x_i\leq\sup_{P\subset Q}\left(\sum_{j=1}^{m} m_j'\Delta x_j'\right)=\underline{\int_a^b} f(x)dx\leq\overline{\int_a^b} f(x)dx=\inf_{P\subset Q}\sum_{j=1}^{m} M_j'\Delta x_j')$$

$\leq\sum_{i=1}^{n} M_i\,\Delta x_i$, where $Q=\{a=x_0',\cdots,x_m'=b\}$

and

$m_j'=\sup\{f(x)\mid x\in[x_{j-1}',\ x_j']\}$,

$M_j'=\inf\{f(x)\mid x\in[x_j'_{-1}, x_j']$, or

$$0\leq\left|\overline{\int_a^b} f(x)dx-\underline{\int_a^b} f(x)dx\right|<2\varepsilon.$$

Since ε is arbitrary, we have

$$\alpha=\underline{\int_a^b} f(x)\,dx=\overline{\int_a^b} f(x)\,dx.$$

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