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A Note in an Inverse System

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〈**Abstract**〉

The purpose of this parper is to find a universal element in the category $\bar{C}$. $Ob(\bar{C})$ consists of pairs$(A, (f_i))$ where $A \varepsilon Ob(\bar{a})$ ($\bar{a}$ is also a category whose objects are topological rings and morphisms are continuous ring homomorphisms) and $f_i: A_i \to A$, $i \varepsilon I$; continuous ring homomorphism, such that for all $i \geqq j$ the following diagram is commutative.

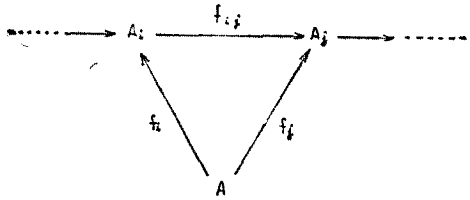

Fig. 1

Inverse System의 연구

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〈요 약〉

우리는 Category에서 Universally Repelling과 Universally Attracting의 성질을 이용하여 중요한 대수적인 성질과 위상적인 성질을 얻어 냈다. 그런데 본 Paper에서는 Continuous Ring Homomorphism을 갖는 Topological Ring의 Category에서 새로운 Category $\bar{C}$를 다음과 같이 만들어 $\bar{C}$에서의 Universally Attracting한 Object가 본래의 Category의 Inverse Limit가 된다는 것을 얻었다.

Category $\bar{C}$의 Object들은 $(A, (f_i))$로 되어 있으며 A는 Category $\bar{a}$의 Object이고 (f_i)는 Continuous Ring Homomorphism의 Family이다.

그러면서 다음 Diagram을 Commutative하게 한다.

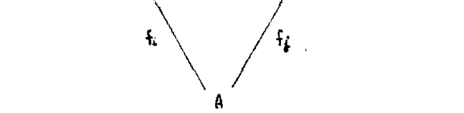

Fig. 2

여기서는 Category $\bar{a}$의 Object는 Topological Ring이고 Morphism들은 Continuous Ring Homomorphism들이다.

I. Introduction

By a free group determined by S, We shall mean a universal element in a category $\bar{C}$ whose objects are the maps of S into groups and if $f: S \to G$ and $f': S \to G'$ are two objects in this category $\bar{C}$, We define a morphism from f to f' to be a homorphism $\varphi: G \to G'$ such that $\varphi \cdot f = f'$

Given a family of objects $\{A_i\}_{i \varepsilon I}$ in $\bar{a}$ a product for this family consists of $(P, \{f_i\}_{i \varepsilon I})$ where P is an object in $\bar{a}$ and $\{f_i\}_{i \varepsilon I}$ is a family of morphisms

$$f_i : P \to A_i$$

satisfying following condition: Given a family of morphisms

$$g_i : C \to A_i$$

there exists a unigue morphism $h: C \to P$ such that $f_i \cdot h = g_i$ gor all $i \varepsilon I$ (P is a universal attracting). Similarily coproduct for the family $\{A_i\}_{i \varepsilon I}$ in $\bar{a}$ is a universal repelling if we reverse the above arrows.

In a category $\bar{C}$, $Ob(\bar{C})$ consists of pairs $(A, (f_i))$ where $A \varepsilon Ob(\bar{a})$ ($\bar{a}$ is also a category whose objects are topological rings and morphisms are continuous ring homomorphisms) and $f_i : A_i \to A$, $i \varepsilon I$; continuous ring homomorphism, such that for all $i \geqq j$, $f_j f_{ij} = f_i$

Then we can find a universal element in $\bar{C}$

II. Preliminaries

In this section, We review the basic concepts relating to the theorem and introduce some notations that will be used in our subsequent development.

Definition 1. A category $\bar{a}$ consists of a collection of objects $Ob(\bar{a})$: and for two objects $A, B \ \varepsilon Ob(\bar{a})$ a set Mor (A, B) called the set of morphisms of A into B; and for three objects $A, B, C \ \varepsilon \ Ob(\bar{a})$ a law of composition

$$\text{Mor } (B, C) \times \text{Mor}(A, B) \to \text{Mor}(A, C)$$

satisfying the following axioms.

Cat. 1. Two set Mor (A, B) and Mor (A', B') are disjoint unless $A = A'$ and $B = B'$, in which case they equal. Cat. 2. For each object A of $\bar{a}$ there is a morphism $id_A \varepsilon$ Mor (A, A) which acts as left and right identity for the elements of Mor (A, B) and Mor (B, A) respectively, for all objects $B \varepsilon Ob(\bar{a})$

3. The law of composition is associative (when defined) *i.e.* Given $f \varepsilon$ Mor (A, B), $g \varepsilon$ Mor (B, C) and $h \varepsilon$ Mor (C, D). then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

for all objects A, B, C, D of $\bar{a}$

Here we write the composition of an element g in Mor (B, C) and an element f in Mor (A, B) as $g \circ f$ to suggest composition of mappings.

Definition 2. Let X be a set. A topology in X is a family $\mathcal{T}$ of subets of X that satisfies:

1) Each union of members of $\mathcal{T}$ is also a member of $\mathcal{T}$

2) Each finite intersection of members of $\mathcal{T}$ is also a member of $\mathcal{T}$

3) and X are members of $\mathcal{T}$. A couple $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ in X is callec a topological spade.

Definition 3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be space. Amap $f: X \to Y$ is called continuous if the inverse image of each open set in Y is open in X.

Definition 4. A ring A is a set, together with two laws of composition called multiplication and addition respectively, and written as a product and as a sum respectively, satisfying the following conditions:

RI 1. With respect to addition, A is a commutative group.

RI 2. The multiplication is associative, and has a unit element.

RI 3. For all x, y, z, ε A we have

$(x+y)z=xz+yz$ and $z(x+y)=zx+zy$

As usual, we denote the unit element for addition by 0, and the unit element for multiplication by 1.

Definition 5. By a ring-homomorphism one means a mapping $f: A \to B$ where A, B are rings, and such that f is a monoid-homomorphism for the multiplcative structures on A and B, and a momnoid-homomorphism for the additive structures. In other words, f must satisfy:

$$f(a+a')=f(a)+f(a'),\quad f(aa')=f(a)f(a')$$
$$f(1)=1 \qquad f(0)=0$$

for all $a, a' \varepsilon A$

Definition 6. A topological ring is a set A which carries a ring structure and a topology sctisfying the following axions:

(AT_1) The mapping $(x,y) \to x+y$ of $A \times A$ into A is continuous.

(AT_2) The mapping $x \to -x$ of A into A is continuous

(AT_3) The mapping $(x,y) \to xy$ of $A \times A$ into A is continuous.

Definition 7. Let I be given a relation of partial ordering in I. Namely for some pairs (i,j) we have a relation $i \geqq j$ satisfying conditions:

For all i,j,k in I, we have $i \geqq i$; if $i \geqq j$ and $j \geqq k$ then $i \geqq k$; if $i \geqq j$ and $j \geqq i$ then $i=j$.

I is said directed if given j,k, there exists i such that $i \geqq j$ and $i \geqq k$.

Definition 8. Let I be a partial ordered set and $\{X_i : i \varepsilon I\}$ be a family of spaces indexed by I. For each pair of indices i,j satisfying $i \geqq j$, assume that there is given a continuous map $f_{ij}: X_i \to X_j$ and that these maps satisfy the following condition: If $i \geqq j \geqq k$, then $f_{ik} \circ f_{ij}$. Then the family $\{X_i : f_{ij}\}$ is called an inverse system over I with X_i and connecting maps f_{ij}

Definition 9. Let $\{X_i : f_{ij}\}$ be an inverse system over I. Form $\Pi\{X_i : i \varepsilon I\}$ and for each α let P_α be its projection onto the αth factor.

The subspace

$\{x \varepsilon \prod_i X_i$: any $i,j \varepsilon I$: $(i \geqq j) \Rightarrow (P_j(x)=f_{ij}P_i(x))\}$ is called the inverse limit space of the system and is denoted by $\varprojlim X_i$

Definition 10. Let $\bar{C}$ be a category. An object P of $\bar{C}$ is called universally attracting if there exists a unique morphism of each object of $\bar{C}$ into P, and is called universally repelling if for every object of $\bar{C}$ there existe a unique morphism of P into this object.

Ⅲ. Theorem

Let I be directed set. Let $\bar{C}$ be a category, and $\{A_i\}$ a family of objects in $\bar{a}$. For each $i \varepsilon I$, A_i is a topological ring. For each pair i,j such that $i \geqq j$, assume given morphism $f_{ij}: A_i \to A_j$

such that, whenever $i \geqq j \geqq k$, we have $f_{jk} \cdot f_{ij} = f_{ik}$ and $f_{ii}=\mathrm{id}$ and for all $i,j \varepsilon I, f_{ij}$ is a continuous ring homomorphism.

Then we can make a catogory $\bar{C}. Ob(\bar{C})$ consists of pairs $(A,(f_i))$ where $A \varepsilon Ob(\bar{a})$and (f_i) is a family of continuous ring homomorphisms $f_i: A \to A_i$, $i \varepsilon I$, such that for all $i \geqq j$ the following diagram is commutative

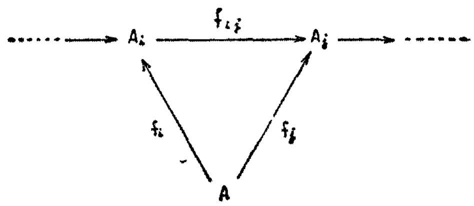

Fig. 3

and then it can be shown that the universally attracting of the category $\bar{C}$ is a direct limit for the family $\{f_{ij}\}$

proof: First $\bar{C}$ is reexamined, $\bar{C}=\{(A,(f_i))$, $A \varepsilon a$, $f_i: A \to A_i$ such that for $i \geqq j$, $f_{ij} \circ f_i = f_j\}$

f is an element of Mor $((A,(f_i))$, $(B,(g_i))$ for $(A,(f_i))$, $(B,(g_i))$ $\varepsilon \bar{C}$ iff f is continuous ring homomorphism and $f: A \to B$ such that $g_i \circ f = f_i$. Now we show that $\bar{C}$ is a category.

1) To show that for each $(A,(f_i))$ in $\bar{C}$ there is a morphism $1_{(A,(f_i))}$ ε Mor $((A,(f_i)),(A,$

$(f_i))$, it is sufficient to take $1_{(A,(f_i))} \equiv 1_A$ such that $1_A: A \rightarrow A$ since commutative diagram

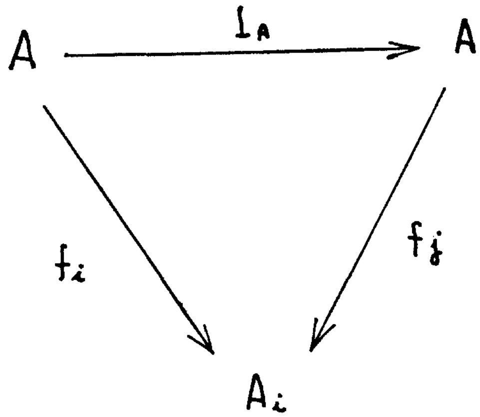

Fig. 4

shows that $f_i \circ 1_A = f_i$

2) To show that $f \varepsilon$ Mor $((A,(f_i)), (B(g_i)))$ and $g \varepsilon$ Mor $((B(g_i)), (C,(h_i)))$ imply $g \circ f \varepsilon$ Moı $((B,(f_i)), (C,(h_i)))$ *i.e* Mor $((B,(g_i)), (C,(h_i))) \times$ Mor $((A,(f_i)), (B,(g_i))) \rightarrow$ Mor $((A,(f_i)), (C,(h_i)))$

It is sufficient to show that $h_i(g \circ f) = f_i$. Btu it is clear from the commutative diagram

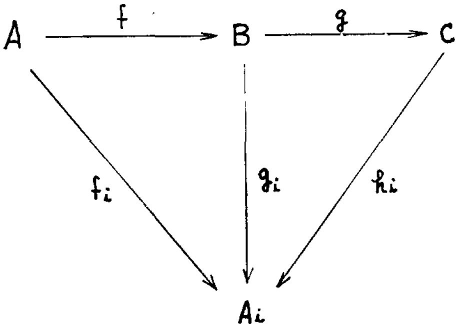

Fig. 5

3) It is clear that the law of composition is associative ($i \cdot e$ $(hog)of = ho(gof)$) As a usual notation

$\varprojlim A_i = \{(x_i);\ x_i \varepsilon A_i,\ P_j((x_i)) = f_{ij}P_i((x_i))$ *for* any $i \geqq j\}$

To show that $(\varprojlim A_i, (P_i))$ is an element of $\bar{C}$ and that $\varprojlim A_i$ is a topological ring, first of all we must show that it is a ring.

RI 1. (definition 4)

$$\begin{aligned} f_{ij}P_i((x_i)-(y_i)) &= f_{ij}(P_i((x_i)) - P_i((y_i))) \\ &= f_{ij}P_i((x_i)) - f_{ij}P_i((y_i)) \\ &= P_j((x_i)) - P_j((y_i)) \\ &= P_j((x_i)-(y_i)) \end{aligned}$$

Hence $(x_i)-(y_i)$ is an element of $\varprojlim A_i$.

Therefore $\varprojlim A_i$ is a additive group •

RI 2 and RI 3 of definition 4 are clear from the definition.

To show AT_1 of definition 6. In the commutative diagram

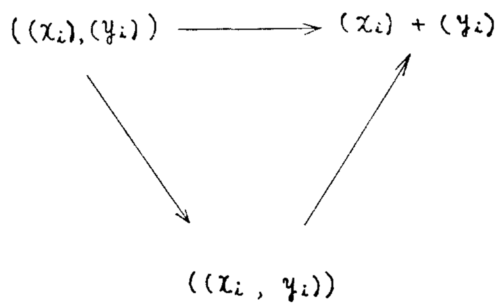

Fig. 6

for any $(x_i), (y_i) \varepsilon \varprojlim A_i$

Since cannonical mapping $((x_i), (y_i)) - ((x_i, y_i))$ is continuous ring homomorphism, and since $((x_i, y_i)) \rightarrow ((x_i + y_i))$ is continuous ring homomorphism, $((x_i), (y_i)) \rightarrow (x_i) + (y_i) = ((x_i + y_i))$ is continuous ring homomorphism.

AT_2 and AT_3 of definition 6 can be attained similarily.

Also since each P_i, $i \varepsilon I$, is projective ring homomorphism, P_i is continuous ring homomorphism.

We will show that $\varprojlim A_i$ is universally attracting. Suppose that $(A,(f_i)$ is an element of $\bar{C}$. In the diagram

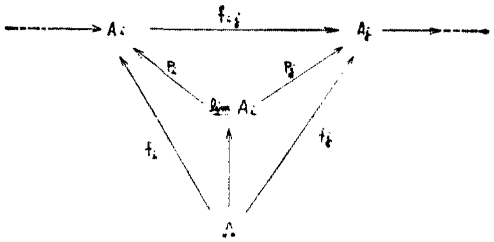

Fig. 7

if we define $f: A \to \varprojlim A_i$ such that $P_i f = f_i$ for each $i \varepsilon I$, then we must show that f is well defined.

Since $f_{ij}P_i(f(x)) = f_{ij}(f_i(x))$
$= f_j(x)$
$= P_j f(x)$,

$f(x)$ is an element of $\varprojlim A_i$. Therefore f is well defined.

If $f_i = P_i f$ is continuous ring homomorphism and if P_i is continuous ring homomorphism, f is continuous ring homomorphism. The proof is as follows:

For all $x_i, y_i \varepsilon A$

$$P_i f(x_i - y_i) = f_i(x_i - y_i)$$
$$= f_i(x_i) - f_i(y_i)$$
$$= P_i f(x_i) - P_i f(y_i)$$
$$= P_i(f(x_i) - f(y_i))$$

$i \cdot e$ $f(x_i - y_i) = f(x_i) - f(y_i)$

Similarily $f(x_i \cdot y_i) = f(x_i) f(y_i)$

$f(1) = 1$, $f(0) = 0$

and the continuity of f is clear. Hence f is an element of Mor $((A, (f_i)), (\varprojlim A_i, (P_i)))$

To show the uniqueness of f, if g is another element of Mor $((A, (f_i)), (\varprojlim A_i, (P_i)))$ then $f = g$. Since for any $x \varepsilon A$, $p_i(g(x)) = f_i(x) = p_i(f(x))$

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