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A Note on Hyperinvariance

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〈**Abstract**〉

In this paper, several sufficient conditions for hyperinvariance and three hyperinvariant subspaces are introduced.

If A is a reductive operator, then A can be written as a direct sum $A_1 \oplus A_2$ where A_1 is normal, A_2 is reductive and all the invariant subspaces of A_2 are hypervariant.

More generally, if A be an operator, there exist two types of hyperinvariant subspaces $\sigma_A(F)$ and S_A (b).

초불변공간의 종류에 관하여

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〈요 약〉

본 논문에서는 무한복소 Hilbert 공간에서 한 유계선형변환의 불변부분공간들이 초불변부분공간이 될 충분조건과 세가지 유형의 초불변부분공간을 소개하였다.

I. Introduction

A bounded (or continuous) linear transformation from a complex Hilbert space into itself is called an operator on H.

Let S and T be operators on H. An operator S commutes with T means $ST=TS$.

Let M be a closed subspace of H and T be an operator on H. We say that M is an invariant subspace of T if $TM \subset M$, and M is hyperinvariant subspace of T(or T has a hyperinvariant subspace M), if $SM \subset M$ for all operators S on H that commute with T.

An operator on a separable Hilbert space is reductive if every subspace invariant for the operator also reduces it.

The family of all invariant subspaces of T on H will be denoted by Lat T and the family of all hyperinvariant subspaces of T by Hyperat T. We denote by $\{A\}'$ the set of all operators that commute with A.

Let L be an abstract lattice. We denote the least upper bound of a and b by $a \vee b$, and the greatest lower bound by $a \wedge b$, for all a, b in L.

A lattice L is called distributive if L satisfies the following conditions:

(1) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

for all a, b, c in L.

The purpose of this paper is to consider the existence condition of nontrivial hyperinvariant subspaces and to introduce several types of hyperinvariant subspaces.

II. Three Types of Hyperinvariant Subspaces

Of course a scalar operator $T=cI$ has only the trivial hyperinvariant subspaces $\{0\}$ and H.

When H is finite-dimensional, a nonscalar operator has nontrivial hyperinvariant subspaces.

Theorem 1. On a finite-dimensional complex linear space, all invariant subspaces of a linear transformation are hyperinvariant if and only if its lattice of invariant subspace is distributive.

Proof. By L. Brickmann and P. A. Fillmore (2)

The following result shows that certain invariant subspaces must be hyperinvariant solely by virtue of their position in the lattice of all invariant subspaces.

Theorem 2. Let S be a countable family of invariant subspaces for an operator T with the property that for any invariant subspaces $M\in S$ and $N\notin S$, either $M\supset N$ or $N\supset M$.

Then S consists of hyperinvariant subspaces.

Proof. This result is due to Rosenthal and Stampfli (4).

Let A be an operator which commutes with T and M be an invariant subspace of T in S.

We have to show $AM\subset M$.

At first it is readily known that the subspace $(A-\lambda I)M$ for all $|\lambda|>\|A\|$ is an invariant subspace of T.

If $(A-\lambda I)M\notin S$ for some $|\lambda|>\|A\|$, then $M\subset(A-\lambda I)M$ or $(A-\lambda I)M\subset M$ by the hypothesis, so that $(A-\lambda I)^{-1}M\subset M$ or $(A-\lambda I)M\subset M$, and in either case $AM\subset M$.

If $(A-\lambda I)M\in S$ for all $|\lambda|>\|A\|$, then $(A-\lambda_1 I)M=(A-\lambda_2 I)M$ for some $\lambda_1\neq\lambda_2$.

Since S is countable,

$$M=(A-\lambda_2 I)^{-1}(A-\lambda_1 I)M$$
$$=(I+(\lambda_2-\lambda_1))(A-\lambda_2 I)^{-1})M.$$

Hence $AM\subset M$. *q.e.d.*

A lattice L is said to be σ-infinitely meet distributive, if L is σ-complete and $a\vee\{\wedge b_n;n\geq1\}=\wedge\{a\vee b_n;n\geq1\}$ for all a, b_n in L.

For any linear transformation T on a finite dimensional complex linear space, Hyperlat T is distributive.

Furthermore, the following theorem shows that σ-infinitely meet distributivity is a sufficient condition for hyperinvariance.

Theorem 3. If Lat T is σ-infinitely meet distributive, then Hyperlat $T=$Lat T.

Proof. This theorem is proved by $W.E.$ Longstaff (7).

Let M and M_i be invariant subspaces of T for all i. We say that M has ascending chain property, if every increasing sequence $M\subset M_1\subset M_2\subset\cdots\cdots$ is stationary. (i.e. there exists n such that $M_n=M_{n+1}\cdots\cdots$)

Dually we can define descending chain property.

The following theorem is another sufficient condition for hyperinvariance.

Theorem 4. If Lat T is distributive and M has ascending chain property (or descending chain property), then M is a hyperinvariant subspace.

Proof. This is already proved by myself (3).

Now, we introduce three kinds of hyperinvariant subspaces.

The following theorem suggests one type of hyperinvariant subspaces.

Theorem 5. If A is a reductive operator, then A can be written as a direct sum $A_1\oplus A_2$ where A_1 is normal, A_2 is reductive, $\{A\}'=\{A_1\}'\oplus\{A_2\}'$ and all the invariant subspaces of A_2 are hyperinvariant.

Proof. This theorem is proved by T. B. Hoover (6).

If A is a completely nonnormal reductive operator, then Lat $A=$Hyperlat A.

In fact, every operator is a nontrivial invariant subspace if and only if every reductive operator is normal (1).

Therefore, it may turn out that there are no nontrivial reductive operators.

However the decompositiion of A in Theorem 5 introduces one type of hyperinvariant subspaces.

On the other hand, suppose that an operator A has the single valued extension property, (i.e., there is no solution $x(\lambda)$ of the equation $(A-\lambda I)x(\lambda)=0$ for all λ in some complex domain D such that $x(\lambda)$ is an analytic function from D to a Banach space B. Define $\theta_A(x)$ to be the set of all λ_0 in the complexes such that the equation$(A-\lambda I)x(\lambda)=x$ is not solvable in any neighborhood of λ_0 with $x(\lambda)$ analytic.

For a closed set F in the complexes, define $\sigma_A(F)$ to be the set of all x in B such that $\theta_A(x)$ does not intersect the complement of F. It is obvious that $\sigma_A(F)$ is a hyperinvariant subspace if $\sigma_A(F)$ is closed.

Let c be a real number. Let $\theta_A(c)$ be the set of all x in B such that $\exp(-ct)\|\exp(At)x\|$ is a bounded function of t, where t ranges over $[0,\infty)$. Given a real mumber b, the intersection of all $Q_A(c)$ with $c>b$ is denoted by $S_A(b)$ Clearly, all $S_A(b)$ are hyperinvariant subspaces for A, if $S_A(b)$ is closed.

Theorem 6. *If* $\sigma_A(F)$ is closed for every closed set F or $S_A(b)$ is closed for all b, and there is a point such that λ_0 is in the spectrum of A, with $Re(\lambda_0)>b$ and S_A(b) nonempty, then A has a nontrivial closed hyperinvariant subspace.

Proof. This is due to Robert M. Kauffman (5).

Let x be in $S_A(b)$. Then λ_0 is not in $\theta_A(x)$ However, if there is no x_0 such that λ_0 is in $\theta_A(x_0)$, then $(A-\lambda_0 I)$ is surjective. In this case, λ_0 is in the point spectrum of A. The null space of $A-\lambda_0 I$ is then a nontrivial closed hyperinvariant subspace. It is nontrivial because if $A=\lambda_0 I$, the hypothesis of the theorem cannot occur.

Thus we need only consider the case that λ_0 is in $\theta_A(x_0)$ for some x_0. Because x_0 is not in $S_A(b)$, $S_A(b)$ is neither $\{o\}$ nor B.

If $S_A(b)$ is closed for all real b, then $S_A(b)$ is a nontrivial closed hyperinvariant subspace. If $\sigma_A(F)$ is closed for every closed set F, let R be an open ball about λ_0 in which the equation $(A-\lambda I)x(\lambda)=x$ is solvable, with $x(\lambda)$ analytic. Let R_1 be the open ball about λ_0 with half the redius of R. Let K be the complement ofR_1. Then x is in $\sigma_A(K)$, but x_0 is not. Thus $\sigma_A(K)$ is a nontrivial hyperinvariant subspace.

q.e.d.

References.

1. J.A. Dyer, E.A. Pedersen and P. Porcelli, An Equivalant formulation of the invariant subspace conjecture; Bull. Amer. Math. Soc. 78 (1972), 1020—1023 *MR* 46#6068
2. L. Brickmann and P.A. Fillmore, The invariant subspace lattice of a linear transformation; Canad. J. Math. 19 (1967), 810—822 *MR*35#4242.
3. Park, Eunsoon, A characterization of hyperinvariance (1978)
4. Peter A. Fillmore, Notes on operator theory: Van Nostrand Reinhold Company, New York (1968).
5. Robert M. Kauffman, Two types of hyperinvariant subspaces; Proc. Amer. Math. Soc. 39 (1973), 553—558.
6. T.B. Hoover, Operator algebras with reducing invariant subspaces; Pacific J. Math. 44(1973), 173—179 *MR* 47 #4026.
7. W.E. Longstaff, A sufficient condition for hyperinvariance Proc. Amer. Math. Soc. 61 (1976), 26—28.