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Nearly $\mathcal{A}$-compact spaces

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〈**Abstract**〉

In this paper a nearly $\mathcal{A}$-compact space is defined and studied a necessary and sufficient condition that a space be nearly $\mathcal{A}$-compact. A necessary condition for an almost regular space to be nearly $\mathcal{A}$-compact and a sufficient condition for a regular space to be nearly $\mathcal{A}$-compact are also studied.

Nearly $\mathcal{A}$-compact 공간

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〈요 약〉

본 연구에서는 nearly $\mathcal{A}$-compact 공간이 정의 되었고, 일반적으로 하나의 공간이 nearly $\mathcal{A}$-compact 공간이 되기위한 필요 충분조건이 검토되었고 또 almost regular 공간이 nearly $\mathcal{A}$-compact 되기위한 필요 조건과 regular 공간이 nearly $\mathcal{A}$-compact 공간이 되기위한 충분조건이 검토되었다.

I. Introduction

V. Ponomarev has introduced the so called $\mathcal{A}$-compact space of which the paracompact space is a special case and the nearly compact space is studied by Larry L. Herrington. In this paper a nearly $\mathcal{A}$-compact space is defined and the characterization of such spaces are given, using system of closed sets. From now on the family of all $(\bar{\alpha})^\circ$, $\alpha\in\mathcal{A}$ $((\bar{V}_\alpha)^\circ,\ V_\alpha\in\alpha)$ will be denoted by $(\bar{\mathcal{A}})^\circ$, $((\alpha)^\circ,\ \alpha\in\mathcal{A})$.

II. Preliminary definitions and theorems

Let $\mathcal{A}$ be the families of open subsets of a given space X such that $(\bar{\mathcal{A}})^\circ$ is a family of open coverings which contains all finite open coverings as a subsystem.

Definition 1. The space X is nearly $\mathcal{A}$-compact if each open covering of X has a refinement $(\bar{\alpha})^\circ$ for $\alpha\in\mathcal{A}$.

Definition 2. A system $\sigma=\{F\}$ of closed sets is called nearly $\mathcal{A}$-tangent if in each $\alpha\in\mathcal{A}$ there is an element $V_\alpha\in\alpha$ such that $(\bar{V}_\alpha)^\circ\cap F\neq\phi$ for all $F\in\sigma$.

Definition 3. A space X is said to be almost-regular if for every point x in X and each neighborhood U of x, there exists a neighborhood V of x such that $V\subset\bar{V}\subset(\bar{U})^\circ$.

From now on we shall suppose that$(\bar{\mathcal{A}})^\circ$ is a directed system (with the natural ordering: $(\bar{\alpha}')^\circ>(\bar{\alpha})^\circ$ if the covering $(\bar{\alpha}')^\circ$ is a refinement of the covering$(\bar{\alpha})^\circ$).

Definition 4. Take in any $\alpha\in\mathcal{A}$ and a set $V_\alpha\in\alpha$. The system $\xi=\{V_\alpha\}$ is called nearly $\mathcal{A}$ thread if for any two $V_\alpha\in\xi$, $V_{\alpha'}\in\xi$, a $V_{\alpha''}\in\xi$ can be chosen with $(\bar{\alpha}'')^\circ>(\alpha)^\circ$, $(\bar{\alpha}'')^\circ>(\bar{\alpha}')^\circ$ (in $(\bar{\mathcal{A}})^\circ$) and $V_{\alpha''}\subseteq(\bar{V}_\alpha)^\circ\cap(\bar{V}_{\alpha'})^\circ$.

Definition 5. The space X has property N_α if for every nearly $\mathcal{A}$-tangent system σ $\{F\}$ the sets $V_\alpha\in\alpha$ with $(\bar{V}_\alpha)^\circ\cap F\neq\phi$ for all $F\subset\sigma$

can be chosen in such a way as to form a nearly $\mathfrak{a}$-thread ("the nearly $\mathfrak{a}$-thread dual to the nearly $\mathfrak{a}$-tangent system σ").

Theorem 1. A space X is nearly $\mathfrak{a}$-compact if and only if each nearly $\mathfrak{a}$-tangent system has a non-void intersection.

Proof. Let $\sigma=\{F\}$ be nearly $\mathfrak{a}$-tangent and $\bigcap_{F\in\sigma} F=\phi$. Then $\bigcup_{F\in\sigma} F^c=X$ which is a covering. If $(\alpha)^\circ$ be a refinement of $\{F^c\}$, then for each $V_\alpha\in\alpha$, there is a $F\in\sigma$ such that $(V_\alpha)^\circ\subset F^c$; i.e. $(V_\alpha)^\circ\cap F=\phi$. But there is a $V_\alpha\in\alpha$ such that $(\overline{V}_\alpha)^\circ\cap F\neq\phi$. for all $F\in\sigma$. This is a contradiction.

For sufficiency; if X is not nearly $\mathfrak{a}$-compact then there is an open covering $\mathscr{U}$ such that for each $\alpha\in\mathfrak{a}$, $(\alpha)^\circ$ cannot be a refinement of $\mathscr{U}$. This says that there exists a $V_\alpha\in\alpha$ for each $\alpha\in\mathfrak{a}$ such that $(\overline{V}_\alpha)^\circ$ cannote be contained in any element of $\mathscr{U}$. This means that $(\overline{V}_\alpha)^\circ\cap U^c\neq\phi$ for all $U\in\mathscr{U}$. Hence the family $\{U^c : U\in\mathscr{U}\}$ of closed sets is a nearly $\mathfrak{a}$-tangent but $\bigcap_{U\in\mathscr{U}} U^c=\phi$.

Lemma. In almost regular space X let $\xi=\{V_\alpha\}$ be a nearly $\mathfrak{a}$-thread and $x\in\bigcap_\alpha V_\alpha$, then all of the open neighborhood $U(x)$ of the point $x\in X$ are among V_α

Proof. In fact, obviously $\bigcap_\alpha (\overline{V}_\alpha)^\circ=\bigcap_\alpha V_\alpha$; for a given open neighborhood $U(x)$ with $\overline{U_1(x)}\subset(\overline{U(x)})^\circ$ and $\alpha_0=\{U(x),\ X-\overline{U_1(x)}\}$. Necessarily $V_{\alpha_0}=U(x)$.

Theorem 2. If an almost regular space X is nearly $\mathfrak{a}$-compact, then both of the following conditions are fullfilled:

(a) the space X has the property $N_{\mathfrak{a}}$,

(b) for each nearly $\mathfrak{a}$-thread $\xi=\{V_\alpha\}$, $(\overline{\xi})^\circ$ has non-void intersection.

Proof. Let X be nearly $\mathfrak{a}$-compact, and $\sigma=\{F\}$ a nearly $\mathfrak{a}$-tangent system. Then $\bigcap_{F\in\sigma} F$ contains a point x by theorem 1.

In any α take an element V_α with $(V_\alpha)^\circ\ni x$. Then the $\xi=\{V_\alpha\}$ thus obtained is a nearly $\mathfrak{a}$-thread. In fact let $V_\alpha\in\xi$, $V_{\alpha'}\in\xi$ be given. Let us choose neighborhoods $U(x)$, $U_1(x)$ of x so that $\overline{U(x)}\subset(\overline{V}_\alpha)^\circ\cap(\overline{V}_{\alpha'})^\circ$, $\overline{U_1(x)}\subset(\overline{U(x)})^\circ$.

First of all let us show that we can choose neighborhood $U(x)$ of x so that $\overline{U(x)}\subset(V_\alpha)^\circ\cap(V_{\alpha'})^\circ$. Since $x\in(\overline{V}_\alpha)^\circ\cap(\overline{V}_{\alpha'})^\circ$, by almost regularity there are neighborhoods $N_1(x)$, $N_2(x)$ of x such that $x\in N_1(x)\subset\overline{N}_1(x)\subset(((\overline{V}_\alpha)^\circ)^-)^\circ=(\overline{V}_\alpha)^\circ$ and $x\in\overline{N_2(x)}\subset N_2(x)\subset(((\overline{V}_{\alpha'})^\circ)^-)^\circ=(\overline{V}_{\alpha'})^\circ$. Let $N_1(x)\cap N_2(x)=U(x)$. Then $\overline{U(x)}=\overline{N_1(x)}\cap\overline{N_2(x)}\subset\overline{N_1(x)}\cap\overline{N_2(x)}\subset(\overline{V}_\alpha)^\circ\cap(\overline{V}_{\alpha'})^\circ$. Take $\alpha_1=\{U(x),\ X-(\overline{U_1(x)})\}$. Take any $(\alpha'')^\circ\in(\mathfrak{a})^\circ$ following $(\alpha)^\circ$, $(\alpha')^\circ$, $(\alpha_1)^\circ$; then $(\overline{V}_{\alpha''})^\circ\subset(\overline{\xi})^\circ$ with $(V_{\alpha''})^\circ\ni x$ and contained in some element of $(\alpha_1)^\circ$, must be contained in $(U(x))^\circ$; therefore $\overline{V}_{\alpha''}\subset(\overline{U(x)})\subset(\overline{V}_\alpha)^\circ\cap(\overline{V}_{\alpha'})^\circ$.

Obviously the nearly $\mathfrak{a}$-thread ξ is dual to σ and the space X has property $N_{\mathfrak{a}}$. Moreover, for any nearly $\mathfrak{a}$-thread $\xi'=\{V_\alpha'\}$, the system $\{V_\alpha'\}$ is a nearly $\mathfrak{a}$-tangent system and $\bigcap_{V_\alpha'\in\xi}(\overline{V}_\alpha')^\circ=\bigcap_{V_\alpha'\in\xi} V_\alpha'$. It follows that $(\overline{\xi})^\circ$ has non-void intersection.

Theorem 3. A regular space X is nearly $\mathfrak{a}$-compact if both of the following conditions are fullfilled:

(a) space X has the property $N_{\mathfrak{a}}$,

(b) for each nearly $\mathfrak{a}$-thread $\xi=\{V_\alpha\}$, $(\overline{\xi})^\circ$ has non-void intersection.

Proof. Let $\sigma=\{F\}$ be a nearly $\mathfrak{a}$-tangent system and $\xi=\{V_\alpha\}$ a dual nearly $\mathfrak{a}$-thread with $x_0\in\bigcap_\alpha(\overline{V}_\alpha)^\circ=\bigcap_\alpha\overline{V}_\alpha\neq\phi$. As $(V_\alpha)^\circ$, i.e. all $(U(x))^\circ$, intersect of all $F_\alpha\in\sigma$, we have $x_0\in\bigcap_{F\in\sigma} F$. Let us show $x_0\in\bigcap_{F\in\sigma} F$, if not ; $x_0\notin\bigcap_{F\in\sigma} F$, i.e. $x_0\in\bigcup_{F\in\sigma} F^c$. Since X is regular, there is a neighborhood $W(x_0)$ such that $x_0\in W(x_0)\subset\overline{W(x_0)}\subset F^c$ for some $F\in\sigma$. Hence $\overline{W(x_0)}\cap F=\phi$. This is a contradiction. Thus X is a nearly $\mathfrak{a}$-compact.

Reference

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