# A fuzzy feebly open set in fuzzy topological spaces

Lee, Je-Yoon · Chae, Gyu-Ihn
Dept. of Mathematics
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〈**Abstract**〉

The object of this paper is to introduce a new form of fuzzy open sets in a fuzzy topological space, named by a fuzzy feebly open set. It is weaker than a fuzzy open set and stronger than a fuzzy semiopen set. We will investigate the properties of fuzzy feebly open sets in a fuzzy topological space.

# Fuzzy 약 개집합에 관하여

이제윤 · 채규인
수 학 과
(1985. 9. 30. 접수)

〈요 약〉

이 논문의 목적은 fuzzy 개집합의 약화된 개집합을 도입하여 그 성질을 연구한다.

## I. Introduction.

The fundamental concept of a fuzzy set, introduced by Zadeh in 1965[12], provides a natural foundation for treating mathematically the fuzzy phenomena which exist in our real world and for building new branches of fuzzy mathematics. Because the concept of fuzzy sets corresponds to the pysical situation in which there is no precisely defined criterion for membership, fuzzy sets are having useful and increasing applications in various fields including probability theory, information theory and pattern recognition.

Since the usual notion of a set was generalization with that of fuzzy sets, the study of a set can be regarded as a special case of fuzzy sets where all fuzzy sets take values 0 and 1 only. C.L. Chang[2] defined fuzzy topological spaces in 1968 by using fuzzy sets as one of applications for fuzzy sets. Since then, several workers continued investigations in fuzzy topological spaces as the study of a generalization of the general topology, that is, a general topology can be regarded as a special case of a fuzzy topology.

In[1], author introduced the stronger and weaker form of a fuzzy open set in fuzzy topological spaces, named by fuzzy regular open sets and fuzzy semiopen sets. In this paper, we will introduce a weaker form of fuzzy open sets, named by fuzzy feebly open sets, and investigate their properties. In section 3, we will study the relations among the stronger and weaker form of the fuzzy open sets.

## II. Fuzzy feebly open sets

Throughout this paper, $X$ and $I$ will denote a nonempty set and the closed unit interval [0, 1] of the real line, respectively.

**Definition 2.1.** Let $I^X$ be a collection of all mappings from $X$ into $I$. A member of $I^X$ is called a *fuzzy set* in $X$. So $g$ is a fuzzy set in $X$ iff $g: X \to [0,1]$ is a function. For every $x \in X$, $g(x)$ is called the *grade of membership* of $x$ in $g$.

Note that if $I$ consists of only the points 0 and 1, then $g$ is just the characteristic function of a subset of $X$, and then $g$ is called a *crisp set* in $X$. Since fuzzy sets are real-valued functions, we will us the existing function operations of $=$, $\leq$, $\wedge$, $\vee$, $+$ and $-$ in order to relate fuzzy sets to other fuzzy sets. The notation $f-g$ means $f(x)-g(x)$ for all $x \in X$ and $f+g$ is defined by $(f+g)(x)=f(x)+g(x)$, with similar definitions for other operations. Note that the crisp set which always takes the value 1 for all $x \in X$ means the set $X$ and the fuzzy set which always takes the value 0 for all $x \in X$ means the fuzzy empty set(constant functions on $X$).

The union $\bigcup g_\alpha$, $\alpha \in A$(an index set) (resp. the intersection $\bigcap g_\alpha$, $\alpha \in A$) of a family $\{g_\alpha | \alpha \in A\}$ of fuzzy sets in $X$ is defined to be the function $\vee g_\alpha$ (resp. $\wedge g_\alpha$). For any two members $f$ and $g$ of $I^X$, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, and in this case $f$ is said to be contained in $g$, or $g$ is said to contain $f$. The complement $g'$ of a fuzzy set in $X$ is $1-g$ defined by $(1-g)(x)=1-g(x)$, for each $x \in X$. By means of properties of $\wedge$ and $\vee$ of real numbers, it is clear that the following De Morgan's law: $(\vee\{g_\alpha | \alpha \in A\})' = \wedge\{g'_\alpha | \alpha \in A\}$.

**Definition 2.2.** Let $X$ be a set and $T(X)$ be a family of $I^X$. Then $T(X)$ is called a *fuzzy topology* on $X$ [2] iff it satisfies the conditions: (a) $0, 1 \in T(X)$, (b) if $g_\alpha \in T(X)$, $\alpha \in A$, then $\vee g_\alpha \in T(X)$ and (c) if $f, g \in T(X)$, then $f \wedge g \in T(X)$.

The pair $(X, T(X))$ is called a *fuzzy topological space* (abb. as *fts*). The elements of $T(X)$ are *fuzzy open sets* in a fts $X$. A fuzzy set $g$ is *fuzzy closed* iff $g' \in T(X)$.

**Definition 2.3.** For a $g \in I^X$ and a fts $X$, the *closure* $cl(g)$ and the *interior* $int(g)$ are defined, respectively, as $cl(g) = \wedge\{f : g \leqq f,\ f' \in T(X)\}$ and $int(g) = \vee\{f : f \leqq g,\ f \in T(X)\}$.

**Lemma 2.4.** Let $X$ be a fts and $f, g \in I^X$. Then

(a) If $f \leq g$, then $cl(f) \leq cl(g)$, $int(f) \leq int(g)$

(b) $cl(cl(f)) = cl(f)$ and $int(int(f)) = int(f)$

(c) $cl(f \vee g) = cl(f) \vee cl(g)$ and $cl(f \wedge g) \leq cl(f) \wedge cl(g)$

(d) $int(f \wedge g) = int(f) \wedge int(g)$ and $int(f) \vee int(g) \leq int(f \vee g)$

(e) $int(1-g) = 1 - cl(g)$ and $cl(1-g) = 1 - int(g)$

**Proof.** We refer to the proof of the Theorem 2.13 in[10].

**Lemma 2.5.** For a family $\{g_\alpha | \alpha \in A\}$ of fuzzy sets in a fts $X$. $\vee cl(g_\alpha) \leq cl(\vee g_\alpha)$. If $A$ is finite, then $\vee cl(g_\alpha) = cl(\vee g_\alpha)$.

**Proof.** It is simple.

In[5], author showed that $\alpha$-sets defined by O. Njastad[7] and feebly open sets due to S. N. Maheshwari[6] are the same set in a ordinary topological space. We will first define a fuzzy feebly open set in a fts as the generalization of the feebly open set in the ordinary topological spaces and investigate its properties.

**Definition 2.6.** Let $X$ be a fts and $g \in I^X$. Then $g$ is called a *fuzzy feebly open set* in $X$ iff $g \leq int(cl(int(g)))$. $g$ is a *fuzzy feebly closed set* in $X$ iff $g'$ is fuzzy feebly open in $X$.

**Theorem 2.7.** (a) Any union of fuzzy feebly open sets is a fuzzy feebly open set.

(b) Any finite intersection of fuzzy feebly open sets is a fuzzy feebly open set.

(c) Any intersection of fuzzy feebly closed sets is a fuzzy feebly closed set.

(d) Any finite union of fuzzy feebly closed sets is a fuzzy feebly closed set.

**Proof.** It follows immediately from Lemma 2.4 and Lemma 2.5.

**Remark 2.8.** Let $FO(X)$ be a collection of all fuzzy feebly open sets in a fts $X$. Then $FO(X)$ is a fuzzy topology on $X$ from (a) and (b) of the above theorem since 0 and 1 is also fuzzy feebly open sets in $X$.

In[9], author defined a neighborhood (abb. as nbd) of a point $x$ of a fts $X$ as a fuzzy set $n$ in $X$ such that there is $g \in T(X)$ such that $g \leq n$ and $n(x)=g(x)>0$. A nbd $n$ of a point $x$ of $X$ is called an open nbd of $x$ iff $n \in T(X)$. Similarly, we define a feeble nbd of a point in a fts.

**Definition 2.9.** A fuzzy set $n_x$ in a fts $X$ is called a feeble nbd of a point $x$ of $X$ iff there exists $g \in FO(X)$ such that $g \leq n_x$ and $n_x(x)=g(x)>0$.

**Theorem 2.10.** If $n_x$ and $p_x$ are feeble nbds of $x$, then $n_x \wedge p_x$ is also a feeble nbd of $x$.

**Proof.** Since $n_x$ and $p_x$ are feeble nbds of $x$, there exist $g, h \in FO(X)$ such that $g \leq n_x$, $h \leq p_x$, $n_x(x)=g(x)>0$ and $p_x(x)=h(x)>0$. Then $g \wedge h \leq n_x \wedge p_x$ and $(n_x \wedge p_x)(x)=(g \wedge h)(x)>0$. $g \wedge h \in FO(X)$ from Remark 2.8. Hence the proof is complete.

**Theorem 2.11.** Let $X$ be a fts. Then $g \in I^X$ is fuzzy feebly open iff, for each $x \in X$ with $g(x)>0$, there is a feeble nbd $n_x \leq g$ such that $n_x(x)=g(x)$

**Proof.** ($\Longrightarrow$) Let $n_x=g$.

($\Longleftarrow$) Let $h=\vee\{n_x \leq g : g(x)>0$. $n_x$ is feeble nbd of $x$ in $X\}$. Then $h$ is fuzzy feely open and $h=g$.

## III. Comparisons.

In this section, we will investigate relations among fuzzy open sets, fuzzy feebly open sets, fuzzy regular open sets and fuzzy semiopen sets due to K.K. Azad[1].

**Definition 3.1.** $g \in I^X$ is called a ***fuzzy regular open set*** (resp. a ***fuzzy regular closed set***) in a fts $X$ iff $g=int(cl(g))$ (resp. $g=cl(int(g))$).

**Definition 3.2.** $g \in I^X$ is called a fuzzy semiopen set in a fts $X$ iff there exists an $h \in T(X)$ such that $h \leq g \leq cl(h)$. The complement of a fuzzy semiopen set is called a fuzzy semiclosed set.

Note that a collection of all fuzzy semiopen sets in a fts $X$ is not a fuzzy topology on $X$ because the intersection of two fuzzy semiopen sets is not, generally, fuzzy semiopen. Refer to Example 3.6.

**Theorem. 3.3.** Every fuzzy open set is fuzzy feebly open.

**Proof.** Let $g$ be a fuzzy open set in a fts $X$. Then $g=int(g)$ [2, p.184]. From Lemma 2.5, we have $g=int(g) \leq int(cl(g)) \leq int(cl(int(g)))$.

**Remark 3.4.** Fuzzy regular open sets are fuzzy feebly open because fuzzy regular open sets are fuzzy open[1] and from Theorem 3.3.

**Theorem 3.5.** Every fuzzy feebly open set is fuzzy semiopen,

**Proof.** Let $g$ be a fuzzy feebly open set in a fts $X$. Then $g \leq int(cl(int(g)))$. Thus the proof follows from that $g \in I^X$ is fuzzy semiopen iff $g \leq cl(int(g))$[1].

The converses of Theorem 3.3, 3.5 and Remark 3.4 are not true, as shown by the following example.

**Example 3.6.** Let $X$ be the closed unit interval $I$ of real numbers. Let $g_1$, $g_2$, $g_3$ $g_4 \in I^X$ defined by:

$$g_1(x)=0, \quad 0 \leq x \leq \frac{1}{2}$$
$$=2x-1, \quad \frac{1}{2} \leq x \leq 1;$$

$$g_2(x)=1, \quad 0 \leq x \leq \frac{1}{4},$$
$$=-4x+2, \quad \frac{1}{4} \leq x \leq \frac{1}{2},$$
$$=0, \quad \frac{1}{2} \leq x \leq 1;$$

$$g_3(x)=0, \quad 0 \leq x \leq \frac{1}{4}$$
$$=\frac{4}{3}x-\frac{1}{3}, \quad \frac{1}{4} \leq x \leq 1;$$

$$g_4(x)=1, \quad 0\leq x\leq \frac{1}{4},$$
$$=-4x+2, \quad \frac{1}{4}\leq x\leq \frac{1}{2},$$
$$=4x-2, \quad \frac{1}{2}\leq x\leq \frac{3}{4},$$
$$=1, \quad \frac{3}{4}\leq x\leq 1:$$

Clearly, $T(X)=\{0, g_1, g_2, g_1\vee g_2, 1\}$ is a fuzzy topology on $X$. Then we have $cl(g_1)=g_2'$, $cl(g_2)=g_1'$, $cl(g_3)=g_2'$, $int(g_3)=g_1$, $int(g_1')=g_2$, $int(g_2')=g_1$, $int(g_4)=g_1\vee g_2$, $cl(g_4)=1$, $cl(g_1\vee g_2)=1$ and $int(g_1\vee g_2)'=0$. Thus, $g_3$ is fuzzy semiopen but not fuzzy feebly open, $g_1\vee g_2$ is fuzzy feebly open but not fuzzy regular open, $g_4$ is fuzzy feebly open but not fuzzy open, and $g_2$ and $g_3$ are fuzzy semiopen but $g_2\wedge g_3$ is not fuzzy semiopen because 0 is the only fuzzy open set contained in $g_2\wedge g_3$ and $cl(0)=0$. $g_1\vee g_2$ is fuzzy open but not fuzzy regular open.

From definitions, theorems and remark in the section 3, we have the following implications: $FRO \Longrightarrow FO \Longrightarrow FFO \Longrightarrow FSO$, where symbols denote, in turn, fuzzy regular open sets, fuzzy open sets, fuzzy feebly open sets and fuzzy semiopen sets.

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