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On α-Convergent Filters

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〈**Abstract**〉

The authors have introduced the concept of α-convergent filters and presented its study.

α- 수렴 필터에 관하여

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〈요 약〉

α- 수렴 필터의 개념을 도입하여 그것에 관하여 연구한다.

I. Introduction

O. Njastad [2] introduced α-sets in topology. A subset S of a topological space X is an α-set if $S\subset \text{Int cl Int } S$, where 'Int' and 'Cl' stand for interior and closure operators respectively. Every open set is an α-set but in general the converse fails [2], Denote by $\alpha(X)$ the class of all α-sets in X. The structure $\alpha(X)$ is a topology on X [2]. The present study is concerned with the role of α-sets in the field of filters.

II. α-Convergent Filters

Definition 1. A subset U of a topological space X is said to be an α-neighbourhood of a point x of X if there is an $S\in\alpha(X)$ such that $x\in S\subset U$,

Denote by $\mathscr{U}_{x(\alpha)}$ the family of all the α-neighbourhoods of x.

Remark 1. Since every open set is an α-set, it is clear that every neighbourhood of a point is an α-neighbourhood of that point. But, in general, the converse is not true. For,

Example 1. Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi,\{a\},X\}$ be the topology on X. Then the set $\{a,c\}\in\mathscr{U}_{x(\alpha)}$ but it is not a neighbourhood of c.

Remark 2. Since the structure $\alpha(X)$ is a topology on X, it is clear that $\mathscr{U}_{x(\alpha)}$ is a filter on X.

Definition 2. A filter $\mathscr{F}$ on a topological space X is said to be α-convergent to a point x of X if $\mathscr{F}\supset\mathscr{U}_{x(\alpha)}$.

Remark 3. It is clear by Remark 1 that if a filter $\mathscr{F}$ on a topological space X, α-converges to a point x then it converges to x, The converse may be false, For, in the space of example 1 $\mathscr{F}=\{X\}$ is a filter on X which converges to c but it does not α-converge to c.

Definition 3. A filter base $\mathscr{B}$ on a topological space X is said to α-converge to a point x of X if the filter whose base is $\mathscr{B}$ α-converges to x.

Proposition 1. A filter $\mathscr{F}$ on a topological

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space X α-converges to a point x of X if and only if every ultrafilter containing $\mathscr{F}$ α-converges to x.

Proof. If $\mathscr{F}$ α-converges to x then evidently every ultrafilter containing $\mathscr{F}$ also α-converges to x. Conversely, let every ultrafilter containing $\mathscr{F}$ α-converges to x. Since $\mathscr{F}$ is the intersection of all the ultrafilters on X finer than $\mathscr{F}$ and so, $\mathscr{F}\supset\mathscr{U}_{x(\alpha)}$.

Proposition 2. Let $\mathscr{B}$ be a filter base on a topological space X. Then $\mathscr{B}$ α-converges to x if every α-neighbourhood of x contains a member of $\mathscr{B}$.

Proof. Suppose $\mathscr{B}$ is α-convergent to x. Then filter on X of base $\mathscr{B}$ α-converges to x. And so every α-neighbourhood of x being a member of this filter contains a member of $\mathscr{B}$. Conversely, if each α-neigbourhood of x contains a member of $\mathscr{B}$ then it belongs to the filter of base $\mathscr{B}$. And so, $\mathscr{B}$ α-converges to x.

Definition 4. A subset B of a topological space X is a coα-set if $X-B\in\alpha(X)$ [1].

Definition 5. In a topological space X, the intersection of all the coα-sets containing a set A, is called the α-closure of A and denoted by $\alpha\text{cl}\,A$. It is a coα-set [1].

Lemma 1. For each α-set U containing x, $U\cap A\neq\phi$ if and only if $x\in\alpha\text{cl}\,A$ [1].

Proposition 3. Let X be a topological space and $A\subset X$. Then $y\in\alpha\text{cl}\,A$ if and only if there is a filter base on A α-converging to y.

Proof. Let $y\in\alpha\text{cl}A$. Then by lemma 1, for each α-neighbourhood U of y, $U\cap A\neq\phi$. So that $\mathscr{B}=\{U\cap A\,|\,U\in\mathscr{U}_{y(\alpha)}\}$ is a filter base on A and clearly it α-converges to y. Conversely suppose that $\mathscr{B}$ is a filter base on A which α-converges to y. Then for each α-neighbourhood U of y there is a $B_\beta\in\mathscr{B}$ such that $B_\beta\cup U$. So every α-neighbourhood of y intersects A. That is, $y\in\alpha\text{cl}\ A$ by lemma 1.

Definition 6. A point x is said to be an α-limit point of a topological space X if every α-neighbourhood of x contains a point of A distinct from x.

Proposition 4. Let A be a subset of a topological space X and let $x\in X$, Then x is an α-limit point of A if and only if there is a filter base on $A-\{x\}$ which α-converges to x.

Proof. Let x be an α-limit point of A. This implies that each $U\in\mathscr{U}_{x(\alpha)}$, $U\cap(A-\{x\})\neq\phi$. Andso, $\mathscr{B}=\{U\cap(A-\{x\})\,|\,U\in\mathscr{U}_{x(\alpha)}\}$ is a base of some filter on $A-\{x\}$, which evidently α-converges to x. Conversely, let $\mathscr{B}$ is a filter base on $A-\{x\}$ which α-converges to x. If $U\in\mathscr{U}_{x(\alpha)}$ then by proposition 2, there is a $B\in\mathscr{B}$ such that $B\subset U$. Since B is a nonempty subset of $A-\{x\}$ it follows that $U\cap(A-\{x\})\neq\phi$, Consequently, x is an α-limit point of A.

Propolsition 5. Let A be a subset of a topological space X. Then $A\in\alpha(X)$ if and only if A belongs to every filter which α-converges to a point of A.

Proof. Let $A\in\alpha(X)$ and let $\mathscr{F}$ be a filter on X which α-converges to $x\in A$. Then $\mathscr{F}\supset\mathscr{U}_{x(\alpha)}$. Since $x\in A$ and $A\in\alpha(X)$ it follows that $A\in\mathscr{U}_{x(\alpha)}$. Andso, $A\in\mathscr{F}$. Conversely, let A belongs to every filter which α-converges to a point of A. Let $x\in A$. Then $\mathscr{U}_{x(\alpha)}$ is α-convergent to x. By hypothesis, $A\in\mathscr{U}_{x(\alpha)}$. Hence there is an α-set U_x such that $x\in U_x\subset A$. Thus, $A=\cup\{U_x\,|\,x\in A\}$ which is an α-set in X. Consequently, $A\in\alpha(X)$.

Definition 7. A topological space X is said to be α-Hausdorff if and only if for any pair of distinct points x, y of X, there exist disjoint α-sets, U and V in X such that $x\in U$, $y\in V$ [1].

Proposition 6. A topological space X is α-Hausdorff if and only if every α-convergent filter on X α-converges to a unique point.

Proof. Let X be an α-Hausdorff space and let $\mathscr{F}$ be a filter on X which α-converges to two distinct points x and y. Then $\mathscr{F}\supset\mathscr{U}_{x(\alpha)}$ and $\mathscr{F}\supset\mathscr{U}_{y(\alpha)}$. Since X is α-Hausdorff, there exist α-sets U, V in X such that $x\in U$, $y\in V$ and $U\cap V=\phi$. But, $U\in\mathscr{U}_{x(\alpha)}$, $V\in\mathscr{U}_{y(\alpha)}$. And so,

$\phi=U\cap V\in\mathscr{F}$ which is a contradiction because $\mathscr{F}$ is a filter. Consequently, $\mathscr{F}$ α-converges to a unique point,

Conversely, suppose that each α-converging filter on X α-converges to a unique point. Let X be not α-Hausdorff. Then there exist two distct points x, y of X such that every α-neighbourhoood containing x and every α-neighbourhood containing y intersect. Therefore the family $\mathscr{U}_{x(\alpha)}\cup\mathscr{U}_{y(\alpha)}$ generates a filter which evidently α-converges to both the points x and y. This is contrary to the supposition. Hence X is an α-Hausdorff space.

Definition 8. Let f be a function of a topological space X into a topological space Y. Then f is said to be α-irresolute if $f^{-1}(B)\in\alpha(X)$ for each $B\in\alpha(Y)$ [1].

Lemma 2. Let f be a function of a topological space X into a topological space X is α-irresolute if and only if for each $x\in X$ the inverse image under f of every α-neighbourhood of $f(x)$ is an α-neighbourhood of x.

Proof. Let f be α-irresolute. Let $x\in X$ and M be an α-neighbourhood of $f(x)$. Then there exists a $H\in\alpha(Y)$ such that $f(x)\in H\subset M$. This implies that $x\in f^{-1}(H)\subset f^{-1}(M)$. Since f is α-irresolute, $f^{-1}(H)\in\alpha(X)$ and hence $f^{-1}(M)$ is an α-neighbourhood of x. Conversely, assume that for each $x\in X$, the inverse image of every α-neighbourhood of $f(x)$ is an α-neighbourhood of x. Let $H\in\alpha(Y)$. If $f^{-1}(H)=\phi$ then it belongs to $\alpha(X)$. If $f^{-1}(H)\neq\phi$ let x be any point of $f^{-1}(H)\neq\phi$ let x be any point of $f^{-1}(H)$. Then $f(x)\in H$. But $H\in\alpha(Y)$ so it is an α-neighbourhood of $f(x)$ in Y. By hypothesis $f^{-1}(H)$ is an α-neighbourhood of x. So that there exists an $A_x\in\alpha(X)$ such that $x\in A_x\cap f^{-1}(H)$. Therefore, $f^{-1}(H)=\cup\{A_x|x\in f^{-1}(H)\}$ which is an α-set in X. Consequently, f is α-irresolute.

Proposition 7. Let f be a function of a topological space X into a topological space Y. Then the following statement are equivalent.

(a) f is α-irresolute

(b) For each $x\in X$ and each filterbase $\mathscr{B}$ on X α-converging to x, the filterbase $f(\mathscr{B})$ on Y α-converges to $f(x)$

(c) For each $x\in X$ and each ultrafilter base $\mathscr{B}$ on X α-converging to x, the ultrafilter base $f(\mathscr{B})$ on Y α-converges

Proof. (a)$\Longleftrightarrow$(b). Let f be α-irresoulte Let $x\in X$ and $\mathscr{B}$ be any filterbase on X α-converging to x. Let N be any α-neighbourhood of $f(x)$. Then by Lemma 2, $f^{-1}(N)$ is an α-neighbourhood of x because f is α-irresolute. Since $\mathscr{B}$ α-converges to x, by proposition 2 there exists $B\in\mathscr{B}$ such that $B\subset f^{-1}(N)$. It follows that $f(B)\subset f(f^{-1}(N))\subset N$. Consequently, every α-neighbourhood of $f(x)$ contains a member of $f(\mathscr{B})$. Hence by proposition 2, $f(\mathscr{B})$ α-converges to $f(x)$.

Conversely, let (b) holds. Let N be any α-set in Y. Let $x\in f^{-1}(N)$. Then $f(x)\in N$, So N is an α-neighbourhood of $f(x)$. Now the α-neighbourhood filter $\mathscr{U}_{x(\alpha)}$ α-converges to x and so by hypothesis $f(\mathscr{U}_{x(\alpha)})$ α-converges to $f(x)$. Hence there is a member $f(M)$ of $f(\mathscr{U}_{x(\alpha)})$ such that $f(M)\subset N$. Since $M\in\mathscr{U}_{x(\alpha)}$, there is an $A_x\in\alpha(X)$ such that $x\in A_x\subset M$. Therefore, $x\in A_x\subset M\subset f^{-1}(f(M))\subset f^{-1}(N)$. And so, $f^{-1}(N)=\cup\{A_x|x\in f^{-1}(N)\}$ which is an α-set in X. Hence, f is an α-irresolute function.

(a)$\Longleftrightarrow$(c), Since every ultrafilter base is a filter base, from (a)$\Longrightarrow$(b) follows that (a) $\Longrightarrow$(c). We now prove that (c)$\Longrightarrow$(a). Assume that f is not α-irresolute. Then by lemma 2, there exists a point $x\in X$ and an α-neighbourhood N of $f(x)$ such that $f^{-1}(N)$ is not an α-neighbourhood of x. This implies that $f^{-1}(N)$ does not contain any α-neighbourhood of x. Then $X-f^{-1}(N)$ intersects every member of $\mathscr{U}_{x(\alpha)}$. Since $\mathscr{U}_{x(\alpha)}$ has the finite intersection property and so $\mathscr{U}_{x(\alpha)}\cup\{X-f^{-1}(N)\}$ also has the finite intersection property. Therefore there exists an ultrafilter $\mathscr{F}$ containing $\mathscr{U}_{x(\alpha)}\cup\{X-f^{-1}(N)\}$. Evidently, $\mathscr{F}$ α-converges to x. But $f(X-f^{-1}(N))\in f(\mathscr{F})$

and evidently, $N\cap f^{-1}(N))=\phi$. Thus there is a member $f(X-f^{-1}(N))$ of $f(\mathscr{F})$ which does not intersect the α-neighbourhood N of $f(x)$. Hence $f(\mathscr{F})$ does not converge to $f(x)$. But this contradicts the hypothesis (c). It follows that f is α-irresolute.

Lemma 3. If a function $f: X\longrightarrow Y$ is open and continous then f is α-irresolute.

Proof. Let $B\in\alpha(Y)$. Then $B\subset$Int cl Int B. Since f is continuos we have, $f^{-1}(B)\subset f^{-1}$ (Int cl Int B)$\subset$Int f^{-1}(cl Int$B$) [3, p. 63]. By the openness of f we have, f^{-1}(cl Int B)$\subset$cl f^{-1} (Int B) [4, (i), p. 13]. Again since f is continuous f^{-1}(Int B)$\subset$Int$f^{-1}(B)$. Thus, $f^{-1}(B)\subset$ Int cl Int $f^{-1}(B)$ Consequently, $f^{-1}(B)\in\alpha(X)$. Hence f is α-irresolute.

Proposition 8. If a filterbase $\mathscr{B}$ on $\prod_{\lambda\in\Lambda} X_\lambda$ α-converges to x then filterbase $p_\lambda(\mathscr{B})$ α-converges to x_λ for each $\lambda\in\Lambda$.

Proof. Each projection p_λ, $\lambda\in\Lambda$ is open and continuous. Therefore for each $\lambda\in\Lambda$, p_λ is α-irresolute by lemma 3. Consequently, Proposition 7 (b) is applicable.

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