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Semisimple Artinian Rings of Fixed Rings.

Lee Tong-su.
Dept of muthematics
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〈Abstract〉

Let G be a finite group of automorphisms of the ring R, and R^G be the ring of fixed points of G in R. Then if R^G is semisimple Artinian, R is semisimple Artinian where R has no nlipotent ideals and no $|G|$ torsion.

고정환의 Semisimple Artinian 환

이 동 수
수 학 과
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〈요 약〉

G가 R 위에 작용하는 automorphism 군일때 G가 고정시키는 R의 모든 원의 집합을 R^G라 하면 R^G가 semisimple Artinian 일때 R도 semisimple Artinian 이다.

I. Introduction

One of the basic questions of noncommutative Galois theory is the relation between a ring R and the fixed ring R^G which is fixed by a group of automorphism G of R. These subject began with some very basic, simple questions asked by George Bergman in 1970. First, he asked whether the fixed subdomain of an Ore domain is again Ore. Second, while trying to answer this question, Bergman and I. M. Issacs asked an even more basic question. If a finite group acts on a domain does there exist nonzero fixed points? From that time many reseachers studied the structure of R and R^G.

Daniel R. Farkas and Robert L. Snider showed that if R^G is neetherian then R is also noetherian where R is semiprime and R has no $|G|$ torsion (3). In this paper we will prove that if R^G is semisimple Artinian then R is semisimple Artinian under some conditions.

Definition 1. (1) The fixed ring of $R : R^G = \{r \in R \mid r^g = r \text{ for all } g \in G\}$.

(2) The ring R is said to have no $|G|$ torsion if $|G|r = 0$ for r in R implies $r = 0$.

(3) An ideal I of R is said to be invariant under G if $I^g = I$ for all $g \in G$.

Bergman and Issac proved the following propositions in 1972.

Proposition 2. Let R be a ring and suppose that G s a finite group acting by automorphisms on R, such that $R^G = \{0\}$. If $R \neq \{0\}$ then R is nilpotent.

Proof. See (1).

Under these proposition we prove some lemmas.

Lemma 3. Let R be a semiprime and have no $|G|$ torsion. If I is G-invariant right ideal of R then $I \cap R^G \neq \{0\}$.

Proof. Since I is G-invariant, G is a group of automorphsims of I. And $R^G \cap I = \{0\}$ implies $I^G = \{0\}$. Thus I is nilpotent by proposition. But R is semiprime implies that R has no nilpotent nonzero ideals. Hence $I^G \neq \{0\}$.

Lemma 4. Let R be a semiprime and have no $|G|$ torsion. Then if R^G has a unit element e, e is a unit for R.

Proof. Let $I = \{r - er \mid r \in R\}$. Then I is a right ideal of R and G-invariant since $e \in R^G$. In fact if $I \neq \{0\}$ then $I \cap R^G \neq \{0\}$. Here if $I \cap R^G \neq \{0\}$ there exists some $y \in R$ such that $y - ey \neq 0 \in R^G$. Then $0 = e(y - ey) = y - ey$ a contradiction. Thus $I = \{0\}$ and so $y = ey$ for all $y \in R$. By similar method we obtain $y = ye$ for all $y \in R$.

II. Main Results

We can prove our theorem by using above lemmas.

Theorem 5. If R^G is semisimple Artinian then R is semisimple Artinian where R has no $|G|$ torsion.

Proof. At first we will prove that R is semisimple. Let $J(R)$ be the Jacobson radical of R. Since image of any automorphism of a quasi regular elements in R is again a quasi regular, $J(R)$ is invariant under any automorphism of R. Thus $J(R)$ is G-invariant. And $J(R) \neq \{0\}$ implies $J(R) \cap R^G \neq \{0\}$. But if $x \in J(R) \cap R^G$, $x^g = x$ for all $g \in G$ and its quasi inverse $y \in R^G$. Thus $J(R) \cap R^G$ is a quasi regular ideal in R^G. By hypothesis R^G is semisimple, hence $J(R) \cap R^G = \{0\}$. Thus $J(R) = \{0\}$.

Secondly we will prove that R is Artinian. Since R^G is Artinian, we may choose a finite family of maximal right ideals of R, $M_1, M_2, \cdots M_n$ such that $I = R^G \cap \left(\bigcap_{i=1}^{n} M_i\right)$ is a minimal ideal in R^G. If $I \neq \{0\}$ we may choose an idempotent e in R^G such that $I = eR^G$ since R^G is semisimple. Then $(1-e)R$ will be a proper ringt ideal of R. And we can find a maximal right ideal $M_{n+1} \subset R$ containig $1-e$. Since $1-e \in M_{n+1}$, $e \notin M_{n+1}$ hence $R^G \cap \left(\bigcap_{i=1}^{n} M_i\right)$ is properly smaller than I. This is contradiction on the assumption that I is minimal ideal in R^G. Thus we conclude that $I = \{0\}$. Let $M = \bigcap_{i=1}^{n} M_i{}^g$ for all $g \in G$. Then M is G-invariant and $M \subset R^G \cap I = \{0\}$. By lemma 3, $M = \{0\}$. In this case $\{0\}$ is the intersection of finitely many maximal right ideals of R. Thus R has finite composition length as a right module over itself. Hence R is Artinian as a right R-module. Thus R is Artinian. The theorem is proved.

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