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Locally nearly-compact spaces

Chung, In Jae
Dept. of General Education.

〈**Abstract**〉

We define a locally nearly-compact space and give several characterizations of such spaces, some of which make use of filters, nets and a type of convergence we define as *n*-convergence.

국소 nearly-compact 공간

정　　인　　재
교양과정부

〈**요　　약**〉

우리는 국소 nearly-compact공간을 정의하고, 한공간이 국소 nearly-compact공간이 되기 위한 필요충분 조건들을 제시하고, 이들 조건들 중의 몇가지는 filter와 net들이 *n*-convergence하거나 *n*-accumulate한다는 것을 정의하여 이들을 사용하여 증명한다.

I. Introduction

The object of this paper is to introduce a locally nearly-compact space. We give several characterization of such spaces, some of which make use of filters, nets and a type of convergence we define as *n*-convergence. In this paper the method of proof based on the idea of [1] and [2]. A set A is called regular closed, if it is the closure of its own interior or equivalent, if it is a closure of some open set. Throughout this paper, $\bar{N}$ denote the closure of a set N and N° denote the interior of a set N.

II. Preliminary definitions and theorems

Definition 2.1. A space X is locally nearly-compact if for each point $x \in X$ and each open cover $\{U_\alpha | \alpha \in A\}$ of a neighborhood $N(x)$ of x, there exists a finite subcollection $\{U_{\alpha_i} | i=1, 2, \cdots, n\}$ such that $N(x) \subset \bigcup_{i=1}^{n} (\bar{U}_{\alpha_i})^\circ$

Definition 2.2. A space X is nearly-compact if for each open cover $\{U_\alpha | \alpha \in A\}$ of X, there exists a finite subcollection $\{U_{\alpha_i} | i=1, 2, \cdots, n\}$ such that $X \subset \bigcup_{i=1}^{n} (\bar{U}_{\alpha_i})^\circ$

Let a set U be regular open if $(\bar{U})^\circ = U$. In view of the fact that for any open set U, $(\bar{U})^\circ$ is regular open, it follows immediately that the open sets in Definition 2.1 and Definition 2.2 may be replaced with regular open sets and an equivalent definition obtained. We now show the same may be done with Defintion 2.1.

Theorem 2.1. The space X is locally nearly-compact if and only if for each point x in X and regular open cover $\{U_\alpha : \alpha \in A\}$ of a neighborhood $N(x)$ of x, there is a finite subcollection $\{U_{\alpha_i} | i=1, 2, \cdots, n\}$ such that $N(x) \subset \bigcup_{i=1}^{n} (\bar{U}_{\alpha_i})^\circ$

Proof. If X is locally nearly-compact, the condition follows from Definition 2.1. Now suppose the condition holds and let $\{U_\alpha|\alpha\in A\}$ be an open cover of a neighborhood $N(x)$ of x in X. Then $\{(\overline{U}_\alpha)^\circ|\alpha\in A\}$ is a regular open cover of $N(x)$ so there exists a finite subcollection $\{(U_{\alpha_i})^\circ|i=1,2,\cdots,n\}$ such that $N(x)\subset\bigcup_{i=1}^{n}(U_{\alpha_i})^\circ$ which shows X is locally nearly-compact.

Definition 2.3. Let X be a space and let $N(x)$ be a neighborhood of x in X. Let $\mathscr{F}=\{N_\alpha\subset N(x)|\alpha\subset A\}$ be a filter base in $N(x)$. Then $\mathscr{F}$ n-converge to $y\in N(x)$ if for each open set V in X containg y there exists a $N_\alpha\in\mathscr{F}$ such that $N_\alpha\subset(V)^\circ$. The filter base $\mathscr{F}$ n-accumulates to $y\in N(x)$ if for each open set V in X containing y and for each $N_\alpha\in\mathscr{F}$, $N_\alpha\cap(\overline{V})^\circ\neq\phi$.

We list several results concerning n-convergence and n-accumulation whose straightforward proofs are omitted.

Theorem 2.2. Let X be a topological space and $N(x)$ be a neighborhood of x in X. Then

(a) If $\mathscr{F}$ is a filterbase in $N(x)$ such that $\mathscr{F}$ n-converge to $y\in N(x)$, then $\mathscr{F}$ n-accumulates to y

(b) Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two filterbases in $N(x)$ and suppose $\mathscr{F}_2$ is stronger $\mathscr{F}_1$. If $\mathscr{F}_2$ n-accumulates to $y\in N(x)$, then $\mathscr{F}_1$ n-accumulates to $y\in N(x)$.

(c) Let $\mathfrak{M}$ be a maximal filterbase in $N(x)$. Then $\mathfrak{M}$ n-accumulates to $y\in N(x)$ if and only if $\mathfrak{M}$ n-converge to $y\in N(x)$.

Definition 2.4 Let $N(x)$ be a neighborhood of x in X. If D is a directed set and $\phi:D\to N(x)$ is a net in $N(x)$, then

(a) Φ n-converge to $y\in N(x)$ if for each open set $V\subset X$ containing $y\in N(x)$, there exists a $z\in D$ such that $\Phi(T_z)\subset(V)^\circ$ where $T_z=\{z'\in D|z'\geq z\}$.

(b) Φ n-accumulates to $y\subset N(x)$ if each open set $V\subset X$ containing y and for every $z\in D$, $\Phi(T_z)\cap(\overline{V})^\circ\neq\phi$.

If $\Phi:D\to N(x)$ is a net in a neighborhood $N(x)$ of x, the family $\mathscr{F}(\Phi)=\{\Phi(T_z)|z\in D\}$ is a filterbase in $N(x)$ and it is routine to verify that

(1) $\mathscr{F}(\Phi)$ n-converges to $y\subset N(x)$ if and only if Φ n-converges to $y\in N(x)$, and

(2) $\mathscr{F}(\Phi)$ n-accumulates to $y\in N(x)$ if and only if Φ n-accumulates to $y\in N(x)$.

Conversely, every filter base $\mathscr{F}$ in $N(x)$ determines a net $\Phi:D\to N(x)$ such that

(1) $\mathscr{F}$ n-converges to $y\in N(x)$ if and only if Φ n-converges to $y\in N(x)$, and

(2) $\mathscr{F}$ n-accumulates to $y\in N(x)$ if and only if Φ n-accumulates to $y\in N(x)$

Ⅲ. Filterbase and net characterizations of locally nearly-compact spaces.

Theorem 3.1. In a space X the following are equivalent:

(a) X is locally nearly-compact.

(b) For each point $x\in X$ and each regular open cover $\{U_\alpha|\alpha\subset A\}$ of a neighborhood $N(x)$ of x, there exists a finite subcollection $\{U_{\alpha_i}|i=1,2,\cdots,n\}$ such that $N(x)\subset\bigcup_{i=1}^{n}U_{\alpha_i}$.

(c) For each point x in X and each collection of nonemty closed sets $\{F_\alpha|\alpha\in A\}$ such that $(\bigcap_\alpha F_\alpha)\cap N(x)=\phi$ for a neighborhood $N(x)$ of x, there is a finite subcollection $\{F_{\alpha_i}|i=1,2,\cdots,n\}$ such that $(\bigcap_{i=1}^{n}F_{\alpha_i})\cap N(x)=\phi$.

(d) For each point x in X and each collection of nonemty regular closed sets $\{F_\alpha|\alpha\in A\}$, if each finite subcollection $\{F_{\alpha_i}|i=1,2,3,\cdots,n\}$ has the property that $(\bigcap_{i=1}^{n}F_{\alpha_i})\cap N(x)\neq\phi$ for each neighborhood $N(x)$ of x, then $(\bigcap_\alpha F_\alpha)\cap N(x)\neq\phi$.

(e) For each point $x\in X$ and each filterbase $\mathscr{F}=\{N_\alpha|\alpha\in A\}$ in a neighborhood $N(x)$ of x, there exists a $y\in N(x)$ such that $\mathscr{F}$ n-accumulates to $y\in N(x)$.

(f) For each point x in X and maximal filterbase $\mathfrak{M}=\{N_\alpha|\alpha\in A\}$ in a neighborhood $N(x)$ of x there exists a $y\in N(x)$ such that $\mathfrak{M}$ n-converges to $y\in N(x)$.

Proof. (a) if and only if (b) has been shown in theorem 2.1.

(b) implies (c). For each point x and a neighborhood $N(x)$ of x in X let $\{F_\alpha|\alpha\in A\}$ be a collection of regular closed sets such that $(\bigcap_\alpha F_\alpha)\cap N(x)=\phi$. Then $N(x)\subset X-\bigcap_\alpha F_\alpha=\bigcup_\alpha(X-F_\alpha)$. Since $X-F_\alpha$ is regular open for each $\alpha\in A$, the hypothesis of (b) implies there is a finite subcollection $\{X-F_{\alpha_i}|i=1,2,\cdots,n\}$ such that $N(x)\subset\bigcup_{i=1}^n(X-F_{\alpha_i})=X-\bigcap_{i=1}^n F_{\alpha_i}$. It follows that $N(x)\cap(\bigcap_{i=1}^n F_{\alpha_i})=\phi$.

(c) implies(b). For each point $x\in X$ and a neighborhood $N(x)$ of x, let $\{U_\alpha|\alpha\in A\}$ be a regular open cover of $N(x)$. Then $N(x)\subset\bigcup_\alpha U_\alpha$ implies $N(x)\cap(\bigcap_\alpha(X-U_\alpha))=\phi$ Since $X-U_\alpha$ is regular-closed for each $\alpha\in A$, the hypothesis of (c) implies there is a finite subcollection $\{X-U_{\alpha_i}|i=1,2,\cdots,n\}$ such that $N(x)\cap(\bigcap_{i=1}^n(X-U_{\alpha_i}))=\phi$. It follows that $N(x)\subset\bigcup_{i=1}^n U_{\alpha_i}$.

(c) if and only if (d) is clear.

(b) implies (e). For each point x in X and a neighborhood $N(x)$ of x if there exists a filterbase $\mathscr{F}=\{N_\alpha|\alpha\in A\}$ in $N(x)$ such that $\mathscr{F}$ does not n-accumulate to y for all $y\in N(x)$, then for each $y\in N(x)$ there exists a regular open set $U(y)$ and some $N_{\alpha(y)}\in\mathscr{F}$ such that $N_{\alpha(y)}\cap U(y)=\phi$. The collection $\{U(y)|y\in N(x)\}$ is a regular open cover of $N(x)$, so by (b) there exists a finite subcollection $\{U(y_i)|i=1,2,\cdots,n\}$ such that $N(x)\subset\bigcup_{i=1}^n U(y_i)$. Let $N_{\alpha_0}\in\mathscr{F}$ such that $N_{\alpha_0}\subset\bigcap_{i=1}^n N_{\alpha(y_i)}$. Since $N_{\alpha_0}\neq\phi$, there is some $1\leq j\leq n$ such that $N_{\alpha_0}\cap U(y_j)\neq\phi$. This implies $N_{\alpha(y_j)}\cap U(y_j)\neq\phi$ which is a contradiction. Condition (c) follows.

(e) implies (d) For each point x in X and a neighborhood $N(x)$ of x, if a collection of regular-closed sets $\{F_\alpha|\alpha\in A\}$ such that each finite subcollection $\{F_{\alpha_i}|i=1,2,\cdots,n\}$ has the property that $(\bigcap_{i=1}^n F_{\alpha_i})\cap N(x)\neq\phi$ but $(\bigcap_\alpha F_\alpha)\cap N(x)=\phi$. Then the sets $F_\alpha\cap N(x)$, $\alpha\in A$, together with all finite intersections of the form $(\bigcap_{i=1}^n F_{\alpha_i})\cap N(x)$, form a filterbase $\mathscr{F}$ in $N(x)$. By (e), this filterbase n-accumulates to some point $y\in N(x)$. Thus, for each open $U(y)$ containing y and each F_α, $(\overline{U(y)})^\circ\cap(F_\alpha\cap N(x))\neq\phi$. The fact that $F_\alpha\cap N(x)\neq\phi$ for all $\alpha\in A$ and the assumption that $(\bigcap_\alpha F_\alpha)\cap N(x)=\phi$ give the existence of an $\alpha_0\in A$ such that $y\notin F_{\alpha_0}$. It then follows that $y\in X-F_{\alpha_0}$ which implies $(X-F_{\alpha_0})\cap F_{\alpha_0}=\phi$. But this means $\mathscr{F}$ does not n-accumulates. This contradiction gives $(\bigcap_\alpha F_\alpha)\cap N(x)\neq\phi$.

(e) implies (f). For each point $x\in X$ and a neighborhood $N(x)$ of x if $\mathfrak{M}=\{N_\alpha|\alpha\in A\}$ is a maximal filterbase in $N(x)$, then $\mathfrak{M}$ n-accumulates to $y\subset N(x)$ by (e) so that $\mathfrak{M}$ n converges to y by theorem 2.2. (c).

(f) implies (e). For each point $x\in X$ and a neighborhood $N(x)$ of x if $\mathscr{F}=\{N_\alpha|\alpha\in A\}$ be a filter base in $N(x)$, then there exists a maximal filterbase $\mathfrak{M}$ such that $\mathfrak{M}$ is stronger than $\mathscr{F}$. By (f), $\mathfrak{M}$ n-converges to $y\subset N(x)$. Applying parts (a) and (b) of theorem 2.2., we see that $\mathscr{F}$ n-accumulates to y in $N(x)$.

Our knowledge of the n-convergence and n-accumulation of nets imediately gives the theorem.

Theorem 3.3. In a space X the following are equivalent:

(a) X is locally nearly-compact.

(b) For each point x in X and each net in a neighborhood $N(x)$ of x, there is a point $y\subset N(x)$ such that Φ n-accumulates to y.

(c) For each point x in X and each universal net Φ in a neighborhood $N(x)$ of x, there is a point $y\subset N(x)$ such that Φ n-converges to y.

If we replace the neighborhood in theorem 3.3. with the entire space X, the immediate characterizations of nearly-compact spaces are obtained.

Theorem 3.4. In a space X the following are equivalent:

(a) X is nearly-compact.

(b) Each filterbase $\mathscr{F}=\{N_\alpha|\alpha\in A\}$ in X

n-converges to some $x \subseteq X$.

(c) Each maximal filterbase $\mathfrak{M}$ in X n-converges.

Corollary. In a space X the following are equivallent:

(a) X is nearly compact.

(b) Each net in X has a n-accumulation point.

(c) Each universal net in X n-converges.

Reference

1. BOURBAKI, *General topology*, Part I, pp. 57—85, Addison-Wesly, Ontario (1966)

2. LARRY L. Herrington and PAUL E. Long A. M. S, Vol. 52, pp. 417—426 (1975)