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Equivalence of dipole sheet and vortex sheet both of variable strength density*

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<Abstract>

The equivalence of dipole sheet and vortex sheet is considered in this paper. The velocity induced by dipoles distributed on an open surface is shown, by the use of vector differentiation identities, to be superposition of velocities induced by a large number of ring vortices placed on the surface. This expression is then changed to the sum of two terms; one a line integral along the boundary of the open surface and the other a surface integral on the surface, both integrals carrying the implication of velocity induced by vorticity. The derivation of this expression is through the analysis of dipole strength variation across the cells of grid which covers the open surface. An alternative derivation of equivalence through pure mathematical manipulation is also presented.

가변 강도의 dipole sheet와 vortex sheet의 등가성

이동기
조선 및 해양공학과

<요 약>

강도가 일정치 않은 dipole sheet와 vortex sheet의 등가성을 고찰한다. 한 개 곡면 위에 분포된 dipole에 의하여 유도된 속도가 그 곡면 위에 놓인 많은 ring vortex에 의하여 유도

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된 속도들의 중첩과 같음을 보이고 이 중첩을 vorticity 분포에 의한 유도속도임을 나타내는 두 적분 즉 개곡면의 윤곽선을 따르면서 하는 선적분과 개 곡면 위에서의 면적적분의 합으로 변형한다. 이와 같은 결과의 유도는 개 곡면을 덮는 grid의 cell을 생각하여 인접 cell에서의 dipole 강도변화에 대한 해석을 통해서 이루어지며 한편 순수한 수학적 수식조작에 의해서도 동일한 결과가 얻어짐을 보인다.

Notations

x, y, z	;	a Cartesian coordinate system
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	;	the unit vectors in the direction of x, y and z, respectively
μ	;	dipole strength density
$\mathbf{n}$	;	the dipole axis, a unit vector normal to S
S	;	an open surface on which dipole is distributed, the area of S
Sp	;	projection of S to the x-y plane, the area of Sp
C	;	the boundary of S, positively oriented with respect to $\mathbf{n}$
$\mathbf{x}$	;	a position vector
$\mathbf{u}$	;	the velocity induced by the dipole sheet or by the equivalent vortex sheet
s	;	the distance between a field point (x, y, z) and a point (x', y', z') on S
∇	;	the vector differential operator, $\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$
∇'	;	the vector differential operator, $\mathbf{i}\frac{\partial}{\partial x'}+\mathbf{j}\frac{\partial}{\partial y'}+\mathbf{k}\frac{\partial}{\partial z'}$
N	;	the number of cells on S
$\mathbf{x}_{ck}$	;	the position vector of the center of the k-th cell on S

1. Introduction

It is frequently stated[1,2] that a finite open surface on which dipole is distributed with its axis everywhere normal to the surface is equivalent to a sheet vortex. In the case of a uniform dipole strength density, it can be shown without much complexity, as can be found in a standard text[3], that the equivalent sheet vortex is in fact a ring vortex lying on the edge of the surface, the interior vorticity being cancelled out. If the dipole strength density is variable, however, derivation of the useful expression for the velocity which bears implication of being related to the equivalent sheet vortex is not so straightforward. Hess[4] dealt with this problem by showing that the expression for the velocity induced by a vortex of variable strength along the arc of a closed curve together with the sheet vortex on the surface within the closed curve can be manipulated to become identical to the expression which represents the velocity induced by the dipole sheet. As a way of proving the equivalence, this seems not satisfactory because of lack of interconnection between the physics of the problem and the significance of mathematical expressions.

In this paper, the derivation is attempted, therefore, based on the analysis of velocity induced by a small cell, a large number of which the surface is divided into, as is usual with the proving procedure for the problem of the similar sort. In addition, derivation by a formal manipulation which is more concise than Hess' is presented as an alternative.

2. Ring vortex system equivalent to the dipole sheet

Suppose that dipole of continuously differentiable variable strength density denoted by μ is distributed on an open surface S specified by

$$z = g(x, y) \qquad (1)$$

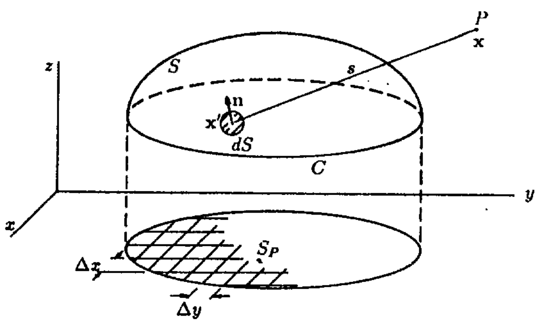


Fig.1 The open surface and its projection covered by grid

Let the axis of dipole, being everywhere normal to the surface, be represented by the unit vector **n**, normal to S and specified by

$$\mathbf{n} = (\mathbf{k} - \mathbf{i}\frac{\partial g}{\partial x} - \mathbf{j}\frac{\partial g}{\partial y}) / [1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2]^{\frac{1}{2}} \qquad (2)$$

C, the boundary of S, is taken to be positively oriented with respect to the vector **n**. The coordinate system is so chosen that Sp which is the projection of S onto the x-y plane has no overlapping part, though overlapping creates no difference to the final results, and that the direction of the z-axis should not be contradictory to eq.(2).

Suppose that the projected area Sp is divided into a large number of rectangular interior cells and triangular boundary cells by grid of straight lines parallel to x- and y- axis. Let this grid be projected back to the surface S. A cell so formed on S will have generally irregular shape and the following relation of area with the cell on Sp

$$\Delta S = [1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2]^{\frac{1}{2}} \Delta Sp + O(\Delta Sp)^{\frac{3}{2}} \qquad (3)$$

The velocity induced at a point $P(\mathbf{x})$ by this dipole sheet is then given[5] by

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi} \nabla \int_S \mu(\mathbf{x}')\mathbf{n}(\mathbf{x}')\cdot\nabla' \frac{1}{s} dS(\mathbf{x}')$$

$$= -\frac{1}{4\pi} \sum_{k=1}^{N} \nabla \int_{S_k} \mu\mathbf{n}\cdot\nabla' \frac{1}{s} dS \qquad (4)$$

where $\nabla' = \mathbf{i}\frac{\partial}{\partial x'} + \mathbf{j}\frac{\partial}{\partial y'} + \mathbf{k}\frac{\partial}{\partial z'}$

$$s = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}$$

S_k ; the k-th cell on S

N ; the number of cells on S.

As $\mu\mathbf{n}$ is invariant with respect to the operator ∇ in the above integrals, the integrand can be changed as the following

$$\nabla\,[\mu(\mathbf{x}')\mathbf{n}(\mathbf{x}')\cdot\nabla'\frac{1}{s}] = -\nabla\times(\mu\mathbf{n}\times\nabla'\frac{1}{s}) \tag{5}$$

Then eq.(4) becomes

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\sum_{k=1}^{N}\nabla\times(\int_{S_k}\mu\mathbf{n}\times\nabla'\frac{1}{s})\,dS \tag{6}$$

The dipole strength density function μ on a cell may be series expanded with respect to its value at any point within the cell, for instance the cell center - the point which projects to the centroid of the corresponding cell on *Sp*. Specifically

$$\mu(\mathbf{x}) = \mu(\mathbf{x}_c) + (\mathbf{x}-\mathbf{x}_c)\cdot(\nabla\mu)_{\mathbf{x}_c} + O(|\mathbf{x}-\mathbf{x}_c|^2)$$

where $\mathbf{x}_c$; the position vector of the cell center (7)

Inserting this expansion into eq.(6), we have

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \frac{1}{4\pi}\sum_{k=1}^{N}\nabla\times\left\{\int_{S_k}[\mu(\mathbf{x}_c) + (\mathbf{x}'-\mathbf{x}_c)\cdot(\nabla'\mu)_{\mathbf{x}_c} + \dots]\,\mathbf{n}\times\nabla'\frac{1}{s}\right\}dS \\ &= \frac{1}{4\pi}\sum_{k=1}^{N}\nabla\times\,[\mu(\mathbf{x}_{ck})\oint_{C_k}\frac{1}{s}\,d\mathbf{x}'\,] + O(S^3/N)^{\frac{1}{2}} \\ &= \sum_{k=1}^{N}\frac{1}{4\pi}\,\mu(\mathbf{x}_{ck})\oint_{C_k}\nabla\frac{1}{s}\times d\mathbf{x}' + O(S^3/N)^{\frac{1}{2}}\end{aligned} \tag{8}$$

where C_k ; the positively oriented closed contour of the k - th cell boundary.

This expression shows that the velocity field associated with a dipole sheet of variable strength density is the same as that associated with large number of ring vortices distributed gaplessly on the sheet, the strength of a ring vortex being identical to the local strength density of the dipole sheet.

3. The surface integral representation of the effect of the interior ring vortices

Although eq.(8) shows the equivalence, as for velocity inducing property, between the dipole sheet and the ring vortex sheet, the expression as it stands is not convenient for practical use. The convenience can be enhanced by segregating the contributions from the boundary of S and those from the inner mesh as follows. Supposing that the grid on S has m strips of cells in the x-direction and dropping the error term, we have, instead of eq.(8), the expression

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\oint_C \mu(\mathbf{x}')\,\nabla\frac{1}{s}\times d\mathbf{x}' + \sum_{i=1}^{m}(\sum_{j=1}^{n_{si}}\frac{1}{4\pi}\Delta\mu_{sij}\int_{C_{sij}}\nabla\frac{1}{s}\times d\mathbf{x}' - \sum_{j=1}^{n_{ci}}\frac{1}{4\pi}\Delta\mu_{cij}\int_{C_{cij}}\nabla\frac{1}{s}\times d\mathbf{x}' \tag{9}$$

with the understanding that $n_{cm}=0$ and that $\Delta\mu_{sij}$ and $\Delta\mu_{cij}$ are defined respectively by

$$\Delta\mu_{sij} = (\mu)_{\mathbf{x}_{cO}} - (\mu)_{\mathbf{x}_{cP}},$$
$$\Delta\mu_{cij} = (\mu)_{\mathbf{x}_{cQ}} - (\mu)_{\mathbf{x}_{cO}}.$$

Refer to Fig.2B for the notations appearing in the above three equations. It is to be noted

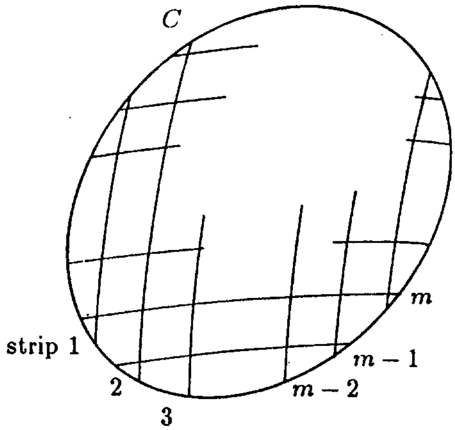


Fig.2A Grid on S

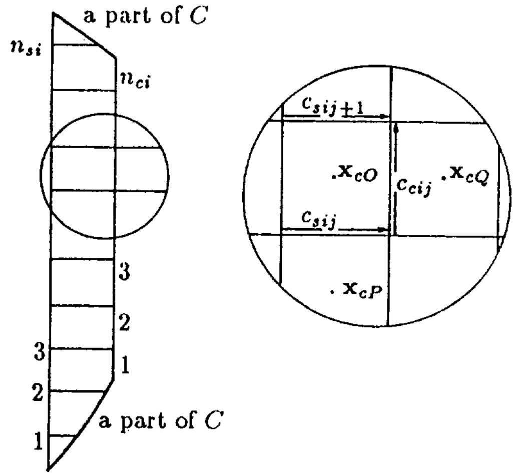


Fig.2B The i-th vortex strip

that only two sub-summations with their member line integrals evaluated along the respective directed line segments shown in the magnified view of Fig.2B are necessary to complete the closed contour integration for each cell on S.

Now consider the interior cells, i.e. cells clear from the boundary. For these cells,

$$\Delta\mu_{sij} = (\nabla\mu)_{\mathbf{x}_{cO}} \cdot [\mathbf{j} + \mathbf{k}(\frac{\partial g}{\partial y})_{\mathbf{x}_{cO}}]\,\Delta y + O(\Delta y^2), \tag{12}$$

$$\Delta\mu_{cij} = (\nabla\mu)_{\mathbf{x}_{cO}} \cdot [\mathbf{i} + \mathbf{k}(\frac{\partial g}{\partial x})_{\mathbf{x}_{cO}}]\,\Delta x + O(\Delta x^2) \tag{13}$$

as the displacement vectors are given respectively by

$$\mathbf{x}_{cO} - \mathbf{x}_{cP} = [\mathbf{j} + \mathbf{k}(\frac{\partial g}{\partial y})_{\mathbf{x}_{cO}}]\,\Delta y + O(\Delta y^2), \tag{14}$$

$$\mathbf{x}_{cQ} - \mathbf{x}_{cO} = [\mathbf{i} + \mathbf{k}(\frac{\partial g}{\partial x})_{\mathbf{x}_{cO}}]\,\Delta x + O(\Delta x^2). \tag{15}$$

Hence, if the number of these cells is denoted by Nc, remembering that one term each from the sub-summations corresponds to one of these cells, we have the following expression for the second term in eq.(9),

$$\frac{1}{4\pi}\sum_{i=1}^{m}\left(\sum_{j=1}^{n_{si}}\Delta\mu_{sij}\int_{c_{sij}}\nabla\frac{1}{s}\times d\mathbf{x}' - \sum_{j=1}^{n_{ci}}\Delta\mu_{cij}\int_{c_{cij}}\nabla\frac{1}{s}\times d\mathbf{x}'\right)$$

$$= \frac{1}{4\pi}\sum_{k=1}^{N_c}\{(\nabla\mu)_{\mathbf{x}_{ck}}\cdot[\mathbf{j} + \mathbf{k}(\frac{\partial g}{\partial y})_{\mathbf{x}_{ck}}]\,\Delta y(\nabla\frac{1}{s})_{\mathbf{x}_{ck}} \times [\mathbf{i} + \mathbf{k}(\frac{\partial g}{\partial x})_{\mathbf{x}_{ck}}]\,\Delta x - (\nabla\mu)_{\mathbf{x}_{ck}}\cdot[\mathbf{i} + \mathbf{k}(\frac{\partial g}{\partial y})_{\mathbf{x}_{ck}}]\,\Delta x(\nabla\frac{1}{s})_{\mathbf{x}_{ck}} \times [\mathbf{j} + \mathbf{k}(\frac{\partial g}{\partial x})_{\mathbf{x}_{ck}}]\,\Delta y\} + O(S^3/N)^{1/2} \tag{16}$$

where $(\nabla\frac{1}{s})_{\mathbf{x}_{ck}} = \nabla[(\mathbf{x} - \mathbf{x}_{ck})\cdot(\mathbf{x} - \mathbf{x}_{ck})]^{-\frac{1}{2}}$

It is to be noted that neglect of the boundary cells brings in the error of order

$$(|\nabla\mu|)_{\max}\,(|\nabla\frac{1}{s}|)_{\max}\,\Delta x \times \text{length of } C = O(\frac{S}{\sqrt{N}}) \tag{17}$$

which is smaller than the error creeping in by the use of eq.(12) and eq.(13) together with the line integrals approximated by the corresponding terms in eq.(16) for the interior cells. However these error bounds do not exceed the error bound specified in eq.(8).

The right-hand side of eq.(16) can be changed to an integral as follows

$$\begin{aligned}
&\frac{1}{4\pi}\int_S \nabla\frac{1}{s}\times\{[\nabla'\mu\cdot(\mathbf{j}+\mathbf{k}\frac{\partial g}{\partial y'})](\mathbf{i}+\mathbf{k}\frac{\partial g}{\partial x'}) \\
&\qquad - [\nabla'\mu\cdot(\mathbf{i}+\mathbf{k}\frac{\partial g}{\partial y'})](\mathbf{j}+\mathbf{k}\frac{\partial g}{\partial y'})\}\,dSp \\
&= \frac{1}{4\pi}\int_S \nabla\frac{1}{s}\times[\nabla'\mu\times(\mathbf{k}-\mathbf{i}\frac{\partial g}{\partial x'}-\mathbf{j}\frac{\partial g}{\partial y'})]\,dSp \\
&= \frac{1}{4\pi}\int_S \nabla\frac{1}{s}\times(\nabla'\mu\times\mathbf{n})\,dS \qquad (18)
\end{aligned}$$

in which use is made of eq.(2) and eq.(3). Inserting this result back into eq.(9) and remembering that the sign changes if s is operated by ∇' instead of ∇, we obtain finally

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\oint_C \mu(\mathbf{x}')\nabla'\frac{1}{s}\times d\mathbf{x}' + \frac{1}{4\pi}\int_S \nabla'\frac{1}{s}\times [\mathbf{n}(\mathbf{x}')\times\nabla'\mu(\mathbf{x}')]\,dS(\mathbf{x}'). \qquad (19)$$

4. Derivation by formal manipulation

The final result eq.(9) can also be derived from eq.(4) formally through the use of vector identities and the Stokes' theorem as follows,

$$\begin{aligned}
\nabla(\mu\mathbf{n}\cdot\nabla'\frac{1}{s}) &= \nabla[\mathbf{n}\cdot\nabla'\frac{\mu}{s} - \mathbf{n}\cdot(\frac{1}{s}\nabla'\mu)] \\
&= \mathbf{n}\cdot\nabla\nabla'\frac{\mu}{s} + \mathbf{n}\times(\nabla\times\nabla'\frac{\mu}{s}) - \\
&\quad \mathbf{n}\cdot\nabla\frac{\nabla'\mu}{s} - \mathbf{n}\times(\nabla\times\frac{\nabla'\mu}{s}) \\
&= \mathbf{n}\cdot\nabla\nabla'\frac{\mu}{s} - \mathbf{n}\cdot\nabla\frac{\nabla'\mu}{s} + \\
&\quad \mathbf{n}\times(\nabla\times\mu\nabla'\frac{1}{s}) \\
&= \mathbf{n}\cdot\nabla\nabla'\frac{\mu}{s} - \mathbf{n}\cdot\nabla\frac{\nabla'\mu}{s}. \qquad (20)
\end{aligned}$$

The first term of the right-hand side can be changed to the other form as shown below

$$\begin{aligned}
\mathbf{n}\cdot\nabla\nabla'\frac{\mu}{s} &= -\nabla\times(\mathbf{n}\times\nabla'\frac{\mu}{s}) + \mathbf{n}(\nabla\cdot\nabla'\frac{\mu}{s}) \\
&= -\nabla\times(\mathbf{n}\times\nabla'\frac{\mu}{s}) + \mathbf{n}(\nabla'\mu\cdot\nabla'\frac{1}{s}). \qquad (21)
\end{aligned}$$

Then eq.(20) becomes

$$\begin{aligned}
\nabla(\mu\mathbf{n}\cdot\nabla'\frac{1}{s}) &= -\nabla\times(\mathbf{n}\times\nabla'\frac{\mu}{s}) + \mathbf{n}(\nabla'\mu\cdot\nabla\frac{1}{s}) \\
&\quad - \mathbf{n}\cdot\nabla\frac{\nabla'\mu}{s} = -\nabla\times(\mathbf{n}\times\nabla'\frac{\mu}{s}) + \\
&\quad \nabla\frac{1}{s}\times(\mathbf{n}\times\nabla'\mu). \qquad (22)
\end{aligned}$$

Inserting this relation into eq.(4), we have

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &= -\frac{1}{4\pi}\int_S [-\nabla\times(\mathbf{n}\times\nabla'\frac{\mu}{s}) + \nabla\frac{1}{s}\times \\
&\quad (\mathbf{n}\times\nabla'\mu)]\,dS = \frac{1}{4\pi}\nabla\times\int_S \mathbf{n}\times\nabla'\frac{\mu}{s}\,dS \\
&\quad + \frac{1}{4\pi}\int_S \nabla'\frac{1}{s}\times(\mathbf{n}\times\nabla'\mu)\,dS = \frac{1}{4\pi}\nabla\times\oint_C \\
&\quad \frac{\mu}{s}\,d\mathbf{x}' + \frac{1}{4\pi}\int_S \nabla'\frac{1}{s}\times(\mathbf{n}\times\nabla'\mu)\,dS. \qquad (23)
\end{aligned}$$

That is

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\oint_C \mu(\mathbf{x}')\nabla'\frac{1}{s}\times d\mathbf{x}' + \frac{1}{4\pi}\int_S \nabla'\frac{1}{s}\times [\mathbf{n}(\mathbf{x}')\times\nabla'\mu(\mathbf{x}')]\,dS(\mathbf{x}') \qquad (24)$$

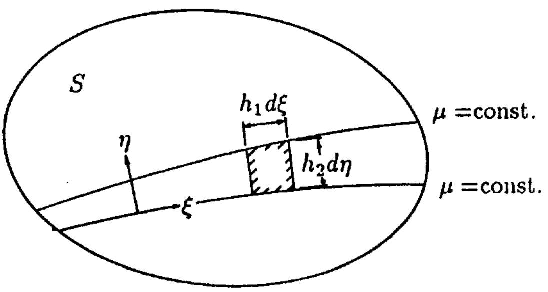
which agrees with eq.(19).

5. Interpretation of the surface integral

The second term of the right-hand side of eq.(19) still means the velocity induced by sheet vortex of variable strength. To show this, let us construct a curvilinear coordinate system (ξ,η), lying on S, taking the ξ-axis along the line μ=constant and the η-axis everywhere orthogonal to this line, positive in the direction of increasing μ. With this coordinate system, since μ is independent of ξ and $\nabla\mu$ is a vector locally parallel to the η-axis

$$\mathbf{n}\times\nabla\mu=\frac{d\mu}{h_2 d\eta}\,\mathbf{e}_\xi$$

where h_2 ; the scale factor for the η coordinate
$\mathbf{e}_\xi$; a unit vector in the direction of the ξ-axis



h_1 ; the scale factor for the ξ-axis

Fig.3 The orthogonal curvilinear coordinates

Therefore

$$\begin{aligned}&\frac{1}{4\pi}\int_S \nabla'\frac{1}{s}\times(\mathbf{n}\times\nabla'\mu)\,dS\\ &=\frac{1}{4\pi}\iint\nabla'\frac{1}{s}\times\frac{d\mu}{h_2 d\eta}\,\mathbf{e}_\xi h_1 h_2\,d\xi d\eta \qquad (26)\\ &=\int\frac{d\mu}{d\eta}\left(\frac{1}{4\pi}\int\nabla'\frac{1}{s}\times\mathbf{e}_\xi h_1 d\xi\right)d\eta\end{aligned}$$

As the inner integral represents the velocity induced by a sheet vortex strip between two neighbouring μ=const. curves, this integral shows the velocity induced by the sheet vortex within S.

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