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A study on Filtering in a Discrete H^∞ Setting

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<Abstract>

The problem of filtering for a discrete time linear system in a discrete H^∞ setting is derived. The measurement noise and process noise have bounded energies. The case with known initial conditions is considered. The approach uses basic quadratic game theory in a discrete domain H^∞ setting.

이산형 H^∞ Norm에서의 필터설계에 관한 연구

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<요 약>

본 논문을 시스템잡음 및 출력잡음이 유한한 에너지를 갖고 있는 시스템에서 H^∞ 이론을 이용하려 부최적화 필터설계를 하기 위한 알고리즘은 개발하는데 이싸. 특히 H^∞ Norm을 계산하기 위하여 부수적으로 수반되는 게임이론을 적용하는것보다 효과적인 필터의 설계를 할 수 있다.

I. Introduction

In this paper, the problem of filtering for a discrete time linear systems in a discrete H^∞ setting is considered. When the time at which an estimate is desired coincides with the last measurement time, the problem is called filtering. Here, the problem of discrete filtering with an H^{∞} performance criterion is considered. The results can be used to solve the associated smoothing and deconvolution problem.

The Kalman filter approach gives the optimal filter algorithm for estimating the states of a linear system when the measurement noise and process noise are zero mean white process with known statistics. The solution to the estimation problem for a linear dynamic system, subjected to exogenous signals whose statistics are known, is well understood in the control literature. When there is significant uncertainty in the power spectral density of the exogenous signals a new measure of performance - the H^{∞} norm - is sometimes useful. Initially introduced by Zames [8], it ensures a more robust design. H^{∞} setting control problem have received considerable attention in the last decade. The control aspect of H^{∞} sense optimization has been studied extensively by Zames [8], Francis and Doyle [2], Bryson [1] and Stoorvogel [7]. Filtering and smoothing results in the continuous time domain have been derived by Nagpal and Khargonekar [5] , with alternative proofs given by Banavar and Speyer [6]. Kim [3] has derived the solution to the problem of prediction in both a full and reduced order discrete H^{∞} setting with unknown initial conditions. Here, the filtering problem for a linear discrete time varying system, considered over a finite time interval in a discrete H^{∞} setting is developed. the approach uses basic quadratic game theory in the discrete time domain, and is similar to the approach used by Banavar [6] and Kim [3].

II. Problem statement for discrete filtering in an H^{∞} setting

In this part, the problem of filtering for discrete time linear system is considered. The measurement noise and process noise are not discrete, white noise vectors, but rather, they are arbitrary discrete disturbances with bounded energy [5]. Only the Riccati equation is derived completely here. Due to algebraic complexity, the filtering portion has been derived using a software tool, MATHEMATICA™ , and the result presented here.

x(j+1) = A(j)x(j) + B(j)w((j) (1)
with linear measurement m(j) = C(j)x(j) + v(j) (2)

where w(j) and v(j) have bounded energies. Define a discrete vector, z(j), which is a linear combination of the states,

z(j) = Lx(j) (3)

where L is selection matrix. Our goal is to estimate z(j) by minimizing the given performance index. The measure of performance is in the form of disturbance attenuation function and can be written as

$$\sup J = \frac{\sum_{j=0}^{N-1} \|z(j) - \hat{z}(j)\|_U^2}{\sum_{j=0}^{N-1} [\ \|w(j)\|_Q^2 + \|v(j+1)\|_R^2]} \qquad (4)$$

where, $\|z(j) - \hat{z}(j)\|_U^2$ is defined as $[z(j) - \hat{z}(j)]^T U [z(j) - \hat{z}(j)]$. The error is defined as $e(j) = z(j) - \hat{z}(j)$

where, $z(j) = Lx(j)$, $\hat{z}(j) = L\hat{x}(j)$ and $e(j) = L[x(j) - \hat{x}(j)]$.

The objective is to ensure that the maximum of the ratio of the energy in the error to the energy in the disturbance is bounded by a positive number γ. It will be seen that this cannot be achieved for arbitrarily small value of γ. The main result is presented in the form of a theorem 1 stated as follows :

Theorem 1 : Let the initial condition be known (without lose of generality, it is assumed that $x(0) = 0$.)

(1) There exists an estimator such that $J < \gamma$ if and only if there exists a positive symmetric matrix $P(j)$ for all j=0, 1, 2, ..., N-1 which satisfy

$$P(j+1) = \phi_{11} P(j) [I - \phi_{21} P(j)]^{-1} \phi_{11}^T + B\theta B^T \qquad (5)$$

where the initial condition is assumed as $P(0) = 0$ and

$$\theta = [\hat{Q}^{-1} + B^T C^T \hat{R}^{-1} CB]^{-1} \qquad (6)$$

$$\phi_{11} = A - B\theta B^T C^T \hat{R}^{-1} CA \qquad (7)$$

$$\phi_{21} = \gamma U - A^T C^T \hat{R}^{-1} CA + A^T C^T \hat{R}^{-1} CB\theta B^T C^T \hat{R}^{-1} CA \qquad (8)$$

(ii) Moreover, if (i) is satisfied, one estimator for which $J < \gamma$ is given as follows :

$$\hat{x}(j+1|j) = A\hat{x}(j) \qquad (9)$$

$$\hat{x}(j+1|j+1) = \hat{x}(j+1|j) + K(j+1)[m(j+1) - C\hat{x}(j)] \qquad (10)$$

where the initial condition is assumed as $\hat{x}(0) = 0$, and

$$K(j+1) = P(j+1) C^T \hat{R}^{-1} \qquad (11)$$

Note that the equation (11) can be described in the same format as the standard discrete Kalman filter gain when $\gamma = \infty$.

Proof :

The performance measure (4) in terms of a game formulation can be shown to be equivalent to

$$\min\max J = \frac{1}{2}\sum_{j=0}^{N-1}\|x(j)-\hat{x}(j)\|_U^2 - \frac{\gamma}{2}\sum_{j=0}^{N-1}\left(\|w(j)\|_{Q^{-1}}^2 + \|v(j+1)\|_{R^{-1}}^2\right) \qquad (12)$$

Since v(j+1) = m(j+1) - C(j+1)x(j+1) from (2), e(j) = L[x(j) - $\hat{x}(j)$] and $U = L^T \hat{U} L$, equation (12) can be defined Hamiltonian as follows :

$$H = \frac{1}{2}\sum_{j=0}^{N-1}\|x(j)-\hat{x}(j)\|_U^2 - \frac{\gamma}{2}\sum_{j=0}^{N-1}\left(\|w(j)\|_{Q^{-1}}^2 + \|v(j+1)\|_{R^{-1}}^2\right) + \lambda^T(j+1)\gamma[\ Ax(j) + Bw(j)] \qquad (13)$$

where $\lambda(j+1)\gamma$ is a Lagrangian multiplier. Optimization for the worst input, for w(j) gives :

$$\frac{\partial H}{\partial w}\Big|_{w=w^*} = 0 \qquad (14)$$

Then we get

$$w^*(j) = \theta B^T[\ \lambda(j+1) + C^T \hat{R}^{-1}[\ m(j+1) - C(j+1)Ax(j)] \qquad (15)$$

The Lagrangian multiplier propagates backwards according the dynamics :

$$\begin{aligned}\gamma\lambda^*(j) &= \frac{\partial H}{\partial x(j)} \\ &= U(x(j)-\hat{x}(j)) + \gamma A^T C^T \hat{R}^{-1}[\ m(j+1) - C(j+1)Ax(j) - C(j+1)Bw(j)] \\ &\quad + \gamma A^T \lambda(j+1)\end{aligned} \qquad (16)$$

or alternatively as

$$\lambda^*(j) = \frac{1}{\gamma}U(x(j)-\hat{x}(j))+A^TC^T\hat{R}^{-1}[m(j+1)-C(j+1)Ax(j)-C(j+1)Bw(j)] + A^T\lambda^*(j+1) \quad (17)$$

By substitution of the value of w(j) and (15) into (17) and some simplification, we can get equation (18).

$$\lambda^*(j) = \phi_{21}x(j) + \phi_{22}\lambda^*(j+1) + \Psi_2 \quad (18)$$

where

$$\phi_{21} = \frac{1}{\gamma}U - A^TC^T\hat{R}^{-1}CA + A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1}CA \quad (19)$$

$$\phi_{22} = A^T - A^TC^T\hat{R}^{-1C}B\theta B^T \quad (20)$$

$$\Psi_2 = -\frac{1}{\gamma}U\hat{x}(j) + [A^TC^T\hat{R}^{-1} - A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1}]\ m(j) \quad (21)$$

Also when $w^*(j)$ is applied, the dynamics for x(j+1) are given as

$$\begin{aligned} x(j+1) &= Ax(j)+B\theta B^T[\lambda^*(j+1)+C^T\hat{R}^{-1}(m(j+1)-C(j+1)Ax(j))] \\ &= (A-B\theta B^TC^T\hat{R}^{-1}CA)x(j)+B\theta B^T\lambda^*(j+1)+B\theta B^TC^T\hat{R}^{-1}m(j+1) \end{aligned} \quad (22)$$

or

$$x(j+1) = \phi_{11}x(j) + \phi_{12}\lambda^*(j+1) + \Psi_1 \quad (23)$$

where

$$\phi_{11} = A - B\theta B^TC^T\hat{R}^{-1}CA \quad (24)$$

$$\phi_{12} = B\theta B^T \quad (25)$$

$$\Psi_1 = B\theta B^TC^T\hat{R}^{-1}m(j+1) \quad (26)$$

The equation (18) and (23) can be written as two point boundary value problem :

$$\begin{bmatrix} x(j+1) \\ \lambda^*(j) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} + \begin{bmatrix} x(j) \\ \lambda^*(j+1) \end{bmatrix} + \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \quad (27)$$

with boundary condition as x(0) = 0 and λ(N)= 0.
Let us define as

$$x^*(j) = x_p(j) + P(j)\lambda^*(j) \quad \text{(28)}$$

in order to express x from (27) in terms of λ. Then, (18) can be described as

$$\lambda^*(j) = \phi_{21}[\ x_p(j) + P(j)\lambda^*(j)]\ + \phi_{22}\lambda^*(j+1) + \Psi_2 \quad \text{(29)}$$

or

$$\lambda^*(j) = [\ I - \phi_{21}P(j)]\ ^{-1}[\ \phi_{21}x_p(j) + \phi_{22}\lambda^*(j+1) + \Psi_2] \quad \text{(30)}$$

Therefore,

$$\begin{aligned} x_p(j+1) + P(j+1)\lambda^*(j+1) = \ & \phi_{11}x_p(j) + \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}[\ \phi_{21}x_p(j) + \phi_{22}\lambda^*(j+1) + \Psi_2] \\ & + \phi_{12}\lambda^*(j+1) + \Psi_1 \end{aligned} \quad \text{(31)}$$

which can be factored as

$$\begin{aligned} & x_p(j+1) - \phi_{11}x_p(j) - \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}\phi_{21}x_p(j) - \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}\Psi_2 \\ & - \Psi_1 + [\ P(j+1) - \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}\phi_{22} - \phi_{12}]\ \lambda^*(j+1) \ = \ 0 \end{aligned} \quad \text{(32)}$$

From (32) we get the vector expression $x_p(j+1)$ and the discrete matrix Riccati equation for the filtering cases as :

$$x_p(j+1) = \phi_{11}x_p(j) + \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}\phi_{21}x_p(j) + \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}\Psi_2 + \Psi_1 \quad \text{(33)}$$

$$P(j+1) = \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}\phi_{11}^T - \phi_{12} \quad \text{(34)}$$

where $\phi_{22} = \phi_{11}^T$ (35)

and ϕ_{11}, ϕ_{21}, ϕ_{12} are given as (24), (19), and (25) respectively.
To prove part (ii), let us rewrite equation (33) as follows :

$$x_p(j+1) = \phi_{11}x_p(j) + \phi_{11}P(j)(I - \phi_{21}P(j))^{-1}[\ \phi_{21}x_p(j) + \Psi_2]\ + \Psi_1 \quad \text{(36)}$$

Now, substituting the values of ϕ_{11}, ϕ_{21}, Ψ_1 and Ψ_2 into (36) we obtain

$$
\begin{aligned}
x_p(j+1) = {} & \phi_{11}x_p(j)+\phi_{11}P(j)(I-\phi_{21}P(j))^{-1}[\ \tfrac{1}{\gamma}U-A^TC^T\hat{R}^{-1}CA \\
& +A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1}CA)x_p(j)-\tfrac{1}{\gamma}U\hat{x}(j) \\
& +(A^TC^T\hat{R}^{-1}-A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1})m(j+1)] \\
& +B\theta B^TC^T\hat{R}^{-1}m(j+1)
\end{aligned}
\tag{37}
$$

The result of simplification and factorization of (37) is shown as follows :

$$
\begin{aligned}
x_p(j+1) = {} & (A-B\theta B^TC^T\hat{R}^{-1}CA)x_p(j)+\phi_{11}P(j)(I-\phi_{21}P(j))^{-1}\{\gamma U[\ x_{p(j)}-\hat{x}(j)] \\
& +(A^TC^T\hat{R}^{-1}[\ m(j+1)-CAx_p(j)]\ -A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1} \\
& [\ m(j+1)-CAx_p(j)]\ \}+B\theta B^TC^T\hat{R}^{-1}m(j+1)
\end{aligned}
$$

(38)

or alternatively as

$$
\begin{aligned}
x_p(j+1) = {} & Ax_p(j)+B\theta B^TC^T\hat{R}^{-1}CA)[\ m(j+1)-CAx_p(j)]\ +\phi_{11}P(j)(I-\phi_{21}P(j))^{-1} \\
& \{\gamma U[\ x_{p(j)}-\hat{x}(j)]\ +A^TC^T\hat{R}^{-1}[\ m(j+1)-CAx_p(j)] \\
& -A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1}[\ m(j+1)-CAx_p(j)]\ \}
\end{aligned}
\tag{39}
$$

Solving the remaining min-max optimization problem with respect to m and $\hat{x}$ as suggested in [6], we get the optimal values, m^* and $\hat{x}^*$ as

$$
m^*(j+1)=CAx_p(j) \quad \text{and} \quad x_p(j)=\hat{x}^*(j). \tag{40}
$$

Letting $\hat{x}^*(j+1|j)=Ax_p(j)$ and $\hat{x}^*(j+1)=x_p(j+1)$, (39) can be written as

$$
\begin{aligned}
\hat{x}^*(j+1) = {} & \hat{x}^*(j+1|j)+[\ \phi_{11}P(j)(I-\phi_{21}P(j))^{-1}(A^T-A^TC^T\hat{R}^{-1}CB\theta B^TC^T\hat{R}^{-1}) \\
& +B\theta B^T]\ C^T\hat{R}^{-1}[\ m(j+1)-CA\,\hat{x}^*(j)]
\end{aligned}
\tag{41}
$$

or alternatively as

$$
\begin{aligned}
\hat{x}^*(j+1) = {} & \hat{x}^*(j+1|j)+[\ \phi_{11}P(j)(I-\phi_{21}P(j))^{-1}\phi_{11}^T+B\theta B^T] \\
& C^T\hat{R}^{-1}[\ m(j+1)-CA\,\hat{x}^*(j)]
\end{aligned}
\tag{42}
$$

where ϕ_{11} and ϕ_{21} are defined in (24) and (19), respectively.

This is similar to the Kalman filter as we can see by defining

$$K(j+1) = [\ \phi_{11}P(j)(I-\phi_{21}P(j))^{-1}\phi_{11}^T + B\theta B^T]\ C^T\ \hat{R}^{-1} \quad \cdots\cdots (43)$$

or

$$K(j+1) = P(j+1)C^T\ \hat{R}^{-1} \quad \cdots\cdots (44)$$

Therefore,

$$\hat{x}(j+1|j) = A\hat{x}(j) \quad \cdots\cdots (45)$$

$$\hat{x}(j+1|j+1) = \hat{x}(j+1|j) + K(j+1)[\ m(j+1) - CA\hat{x}(j)] \quad \cdots\cdots (46)$$

The form of (45) and (46) is obviously similar to the Kalman filter. The gain expressed in (47) is also similar to the Kalman gain, but this fact is less obvious because the Kalman gain is usually expressed in the form :

$$K(j+1) = P(j+1|j)C^T[\ \hat{R} + CP(j+1|j)C^T]^{-1}. \quad \cdots\cdots (47)$$

Therefore, we present the following lemma.

Lemma 1 :
The discrete Kalman filter gain,

$K(j+1) = P(j+1|j)C^T[\ \hat{R}^{-1} + CP(j+1|j)C^T]^{-1}$, can be described as

$$K(j+1) = P(j+1)C^T\ \hat{R}^{-1} \quad \cdots\cdots (48)$$

Proof :
The Kalman filter gain in its standard discrete form can be express as

$$K(j+1) = P(j+1|j)C^T[\ \hat{R} + CP(j+1|j)C^T]^{-1} \quad \cdots\cdots (49)$$

or

$$\begin{aligned} K(j+1) &= P(j+1|j)C^T\ \hat{R}^{-1} - P(j+1|j)C^T\ \hat{R}^{-1} + P(j+1|j)C^T(\ \hat{R} + CP(j+1|j)C^T)^{-1}\hat{R}\ \hat{R}^{-1} \\ &= P(j+1|j)C^T\ \hat{R}^{-1} - P(j+1|j)C^T[\ CP(j+1|j)C^T + \hat{R}]^{-1}CP(j+1|j)C^T\ \hat{R}^{-1} \\ &= P(j+1|j)C^T\ \hat{R}^{-1} - K(j+1)CP(j+1|j)C^T\ \hat{R}^{-1} \\ &= [\ I - K(j+1)C]\ P(j+1|j)C^T\ \hat{R}^{-1} \end{aligned}$$

But, $P(j+1) = [I - K(j+1)\ C]\ P(j+1|j)$ $\cdots\cdots$ (50)

Therefore, $K(j+1) = P(j+1)C^T\ \hat{R}^{-1}$ $\cdots\cdots$ (51)

Since where $\gamma = \infty$, P(j+1) is the same as the Kalman error covariance matrix, the gains are identical when no restrictions are placed on the maximum value of J as indicated by (4). Although when $\gamma = \infty$, the result is identical to the Kalman filter, the interpretation is not the same, i.e, w and v are not white noise. Furthermore, $\hat{R}$ and $\hat{Q}$ are scaling matrices and not covariance matrices.

III. Simulation

Given the discrete linear system of the form by the equations (1) and (2) with the following information :

$$A = \begin{bmatrix} -0.11 & -0.70 & 1.00 \\ -1.00 & 0.30 & 1.00 \\ 1.30 & -1.30 & -0.50 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The covariance of the noise term w and v are Q and R, respectively, with the following values (Q an R known as weighing matrices).

$$Q = 6, \quad R = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The initial state and its statistical information are assumed to be

$$x(0) = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}, \quad E\{x(0)\} = \begin{bmatrix} 4.0 \\ -6.0 \\ -2.5 \end{bmatrix}, \quad var\{x(0)\} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ so that they are needed to compare the Kalman filter results with the H^∞ filter.

Figure 1 shows that a comparison of the actual state x_3 and an estimate of the state x_3 using the H^∞ filtering algorithm. Reference [4] shows the results of considering this example with Kalman filter. This can be compared with our results. In figure 1 we used the white noise for measurement noise and got a very nice estimate of x_3 , although a Kalman filter designed for white noise would be better. Figure 2 is the same as figure 1 except that the white noise is replaced with bounded non white noise. This figure demonstrates that even in unusual noise circumstances we can still get a very nice estimate by using the above mentioned H^∞ filter algorithm. Since our technique does not require any assumption other than that the noise should be bounded, it is sometimes more practical, although the results are not always better than that of the Kalman filter. Finally, figure 3 shows that if one violates the zero mean requirement of the white noise process the results of H^∞ filter is better than that of Kalman filter. In the simulation, the value of $\gamma = 7.3$ was used.

IV. Conclusion

We solved the problems of filtering for a discrete time linear system in a discrete H ∞ setting. The measurement noise and process noise were not necessarily white, but rather bounded energy disturbances. The case with known initial conditions was considered. the approach used basic quadratic game theory in a discrete time domain H^{∞} setting. The results in figure 1, 2 and 3 support the theoretical development. The performance of the H^{∞} setting algorithms depend on the value of γ. When γ is equal to ∞, the results of Kalman filter and H^{∞} filter are the same. Due to a matrix singularity in the Riccati equation, sometimes the problem does not have a solution for a small value of γ.

V. References

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