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Semi T_D-topological spaces

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〈**Abstract**〉

T_0 and T_1 separation axioms are well-known. T_D-axiom introduced by E.C. Aull and W.J. Thron is one of the significant separation axiom between T_0 and T_1. The purpose of this note is to introduce the concept of semi T_D-topological spaces and study their properties. We show the semi T_D-axiom is stronger than the semi T_0-axiom but weaker than the semi T_1-axiom.

Semi T_D-위상 공간에 대하여

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〈요　약〉

우리는 Semi T_D-위상 공간을 도입하여 그 성질을 알아보고 이 위상 공간은 Semi T_0-위상 공간과 Semi T_1-위상공간 사이에 있음을 보인다.

I. Introduction

In 1963 N. Levine [6] defined a set in a topological space to be semi open if there exists an open set O such that $O \subset A \subset cl(O)$, where $cl(O)$ denotes the closure of O. In [2], authors defined a set to be semi closed iff its complement is semi open. A point p is said to be a semi limit point of a set A if each semi open set containing p contains some points of A other than p [4]. Semi closure, semi interior, a semi derived set and etc. are known to be defined in manner analogous to the standard concepts of closure, interior and a derived set. By scl (A), sint(A) and sd(A) we shall denote the semi closure, semi interior and the semi derived set of a set A, respectively. We can also find their definitions in [3].

T_0 and T_1 separation axioms are well known. T_D-axiom introduced by E.C. Aull and W.J. Thron [1], is one of the significant separation axiom between T_0 and T_1. In [7], Maheshwari introduced the semi T_0 and semi T_1 axioms by considering the separation of points through the semi open sets.

The purpose of this note is to introduce the concept of semi T_D-topological spaces in such a way that the semi T_D-axiom is an analogue of the T_D-axiom.

The semi T_D-axiom is found to be stronger than the semi T_0-axiom but weaker than the semi T_1-axiom. Further, the semi T_D-axiom is strictly weaker than the T_D-axiom. However,

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it may fail to be the T_0-axiom, in general. The semi T_D-axiom is profitably used to obtain the following significant result [5].

If X and Y are semi T_D-topological spaces such that both SO(X) and SO(Y) are closed under finite intersections and SO(X) is lattice isomorphic to SO(Y), then X and Y are semi-homeomorphic in the sense of S.G. Crossley [2], where SO(X) and SO(Y) mean the families of all semiopen sets of X and Y, respectively.

Throughout this note, a space means a topological space and iff means if and only if.

II. Semi T_D-axiom

Definition 2.1. A space X is said to be a T_D-space [1] if for each $x \in X$, the derived set of $\{x\}$ is closed.

Definition 2.2. A space X is said to be semi T_0 [7] if for any $x, y \in X$, $x \neq y$, there exists a semi open set G such that either $x \in G$, $y \notin G$ or $x \notin G$, $y \in G$.

Definition 2.3. A space X is said to be semi T_1 [7] if for any $x, y \in X$, $x \neq y$, there exists a semi open set G such that $x \in G$, and a semi open set H such that $x \notin H$, $y \in H$.

It may be mentioned here that every semi T_1-space is semi T_0, but converse may not be true, in general. Now we shall introduce the concept of semi T_D-space.

Definition 2.4. A space X is said to be semi T_D-space if for any $x \in X$, sd($\{x\}$) is semi closed.

Every topological space is not semi T_D. The following example is given for this purpose.

Example 2.5. Let $X = \{a, b, c, d\}$ be a space with $\mathcal{T} = \{0, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $(X, \mathcal{T})$ is not semi T_D.

Following characteriztion of semi T_0-space will be useful in the next discussion.

Theorem 2.6. A space $(X, \mathcal{T})$ is semi T_0 iff for every $x \in X$, sd($\{x\}$) is the union of semi closed sets.

Proof. In a space X, for any $x \in X$, sd($\{x\}$) $= \{y \in X: x \neq y, y \in$ scl($\{x\}$)$\}$. Also, $y \in$ scl($\{x\}$) implies scl($\{y\}$) $\subset$ scl($\{x\}$). If the space X is semi T_0, Then $x \neq y$, $y \in$ scl($\{x\}$) implies $x \notin$ scl ($\{y\}$) and hence scl($\{y\}$) $\subset$ sd($\{x\}$). Thus, in a semi T_0 space X, for any $x \in X$, sd($\{x\}$) $= \cup$ $\{$scl($\{y\}$): $x \neq y$, $y \in$ scl($\{x\}$)$\}$.

Conversely, suppose that the space X is such that for each $x \in X$, sd($\{x\}$) is the union of semi closed sets. For any $x \neq y$, either $y \in$ sd ($\{x\}$) or $y \notin$ sd($\{x\}$). In case $y \in$ sd($\{x\}$), there exists a semi closed set F such that $y \in F \subset$ sd ($\{x\}$). Hence $X - F$ is a semi open set such that $y \notin X - F$, $x \in X - F$. In other case $y \notin$ sd ($\{x\}$), there exists a semi open set G such that $y \in G$, $x \notin G$. Thus the space is semi T_0. The proof is complete.

The next theorem is concerned with the implication relations of semi T_D-axiom between some of other known separation axioms.

Theorem 2.7. (a) Every semi T_D-space is semi T_0.

(b) Every semi T_1 space is semi T_D.

(c) Every T_D-space is semi T_D.

Proof of (a). It is clear in view of Theorem 2.6.

Proof of (b). It follows from the fact that in a semi T_1 space X, for any $x \in X$, $\{x\}$ is semi closed and hence sd($\{x\}$) $= 0$ is semi closed.

Proof of (c). Since, for any x in a space X, scl($\{x\}$) $\subset$ cl($\{x\}$) [3], if the space X is T_D, then, for any $x \in X$, the derived set of $\{x\}$ is closed. Therefore, x can not be a limit point and hence can not be a semi limit point of sd ($\{x\}$). Thus, in order to show that sd($\{x\}$) is semi closed, it is enough to show that each semi limit point y (other than x) is the point of sd($\{x\}$). If $y \neq x$, y is a semi limit point of sd($\{x\}$), then y is also a semi limit point of $\{x\}$. Hence $y \in$ sd($\{x\}$). It follows that every T_D space is semi T_D.

For some of other implication relations, the following example is useful.

Example 2.8. Let $X=\{a,b,c\}$ be a space with $\mathscr{T}=\{\emptyset, \{a\}, X\}$. Then the space X is semi T_D but not T_D. Also X is neither semi T_1 nor T_1.

III. Characterizations of the semi T_D-axiom

Theorem 3.1. A space X is semi T_D iff for each $x\in X$, there exists some semi open set G and semi closed set H such that $\{x\}=G\cap H$.

Proof. Suppose X is semi T_D. Then, for any $x\in X$, sd$(\{x\})$ is semi closed. Taking $G=X-\text{sd}(\{x\})$ and $H=\text{scl}(\{x\})$, we have $\{x\}=G\cap H$ where G is semi open and H is semi closed in X.

Conversely, suppose that the space X is such that each $x\in X$ can be expressed as the intersection of a semi open set and a semiclosed set. Then, for any arbitray point $x\in X$, suppose $\{x\}=G\cap H$ where G is semi open and H is semi closed. Now,

$$\begin{aligned}\text{sd}(\{x\})&=\text{scl}(\{x\})-\{x\}=\text{scl}(\{x\})-(G\cap H)\\&=\text{scl}(\{x\})\cap[(G\cap H)]\\&=\text{scl}(\{x\})\cap[(X-G)\cup(X-H)]\\&=[\text{scl}(\{x\})\cap(x-G)]\cup[\text{scl}(\{x\})\cap(X-H)]\\&=\text{scl}(\{x\})\cap(X-G),\ \text{since scl}(\{x\})\subset H.\end{aligned}$$

Therefore, sd$(\{x\})$ is a semi closed set since any union of semi open sets in a space is semi open.

In a space, if the semi derived set of every set is semi closed, then, obviously, the space is semi T_D. For the converse part, we have the following theorem.

Theorem 3.2. If, in a semi T_D-space, the family of all semi open sets is closed under finite intersection, then the semi derived set of any set is semi closed.

Proof. Suppose that the space X is a semi T_D-space such that intersction of any two semi open sets is semi open. Let A be any subset of X. If sd$(A)=\emptyset$, or has no limit point, then there is nothing to prove. Let x be a semi limit point of sd(A). Because scl(A) $=\text{sd}(A)\cup A$[4], x must belong to either sd(A) or A. In order to show that sd(A) is semi closed, we have to remove the possibility of x belonging to $A-\text{sd}(A)$. For, that if $x\in A$ and $x\notin\text{sd}(A)$, then there exists a semi open neighborhood U of x such that $U\cup A=\{x\}$, x being the semi limit point of sd(A), U must contain some point y of sd(A) other than x. We notethat each such point $y\in\text{sd}(\{x\})$. For, if one such point $y\notin\text{sd}(\{x\})$, then there would exist a semi open neighborhood V of y such that $x\notin V$, and hence $U\cap V$ would be a semi open neighborhood of y such that $(U\cap V)\cap A=\emptyset$. This contradicts the choice of $y\in\text{sd}(A)$. Hence all such $y\in\text{sd}(\{x\})$. Since X is semi T_D, sd $(\{x\})$ is semi closed, but $U\cap(X-\text{sd}(\{x\}))$ is a semi open neighborhood of x disjoint from sd (A). This contradicts the assumption that x is the semi limit point of sd(A). Hence $x\in A$, $x\notin\text{sd}(A)$ is impossible. Thus sd(A) is semi closed.

Corollary 3.3. A space is semi T_D-space iff semi drived set of any subset is semi closed provided the intersection of any two semi open sets is semi open.

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