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Robustness of the LQ State Feedback Control Using Two Different Discrete-Time Representation: the Shift and Euler Operators

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<Abstract>

To quantitatively measure the robustness bounds of the discrete-time LQ state feedback control in the presence of nonlinear perturbations, two theorems are proposed. The robustness bound obtained using the Euler operator converges to the corresponding continuous-time cases both algebraically and numerically. This analysis will help a designer to understand the performance of the discrete-time LQ state feedback control in the presence of nonlinear perturbations, and to select an appropriate sampling interval, which ensures a proper system responses.

Shift & Euler 연산자를 사용한 이산시간 시스템에서의 LQ 제어의 강인성

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<요 약>

비선형 간섭에 노출된 이산시간 LQ 상태 피드백의 강성경계를 정량적으로 측정하기 위하여 2가지의 정리를 제시하였다. 강성경계는 연속시간 시스템에서의 응답으로 변환시키는

Euler 연산자를 이용하여 대수적인 방법과 수치적인 방법으로 얻어진다.

이러한 분석을 통하여 설계자는 비선형 간섭에 노출된 LQ 상태 피드백의 기능을 이해할 수 있고 적절한 샘플링 간격을 설정할 수 있다. 실제계에서 높은 주파수로 샘플링하는 경우 실현하기가 어렵지만, 본 연구에서 제안한 방법은 일반적인 샘플링 주파수로도 Euler 연산자의 강성 정도가 Shift의 그것에 비하여 상당히 높으므로 실제 사용함이 보다 유용하다.

Ⅰ. INTRODUCTION

The numerical superiority over the usual shift operator of digital control laws using the Euler operator has been examined extensively [1]~[3]. These studies show that the Euler operator formulation offers better finite-word-length coefficient representation and less finite-word length rounding error performance in many cases. Moreover, the use of the Euler-operator formulation provides a close correspondence between continuous-time and discrete-time results [4], [5]. Unlike the shift operator, the discrete-time theory based on the Euler operator converges to the appropriate continuous-time results as a sampling rate increases. Such connections provide more flexibility in specifying performance requirements, thereby allowing the digital controllers to be evaluated in a continuous time context. The discrete realization of a continuous-time system is often subject to parameter variations due to finite-word-length effects. Such variations are often very large, and therefore, deteriorate the stability obtainable with the continuous-time LQ state feedback control. This phenomenon becomes more worrisome when the system to be controlled possesses multiple, high-frequency resonances. It is well known that high-frequency resonances in the plant may cause unacceptable sensitivities to disturbances in conjunction with the discretization [6]. Hence, it is important to examine the robustness of the discrete LQ state feedback control in the presence of system uncertainty.

In this regard, an allowable bound in nonlinear perturbations for continuous-time LQ state feedback is extended to the discrete-time LQ state feedback case for easy assessment of its robustness. A quantitative measure of the robustness of the discrete-time LQ state feedback is then used to study the effect of the two different representations: the Euler and the shift operator formulations. It is shown that the discrete-time LQ state feedback using the Euler operator is more robust against nonlinear perturbations than that using the shift operator. Moreover, the resulting response becomes much closer to that of the continuous LQ state feedback as the sampling rate increases, than the shift-operator case. Taking a rotating flexible beam as an example, simulation studies have been performed to assess the allowable level of nonlinear perturbations. It is noteworthy that the said system gives rise to a noncollocated control problem; i.e., the sensors and actuators are placed at different locations on the flexible structure. This introduces unstable zeros, which impose an upper limit on the bandwidth that can be achieved and increase the overall sensitivity to disturbances in the passband of the system. Furthermore, the unmodelled higher modes may reduce the stability margin of the closed-loop system, and thus cause instability associated with the linear-quadratic-Gaussian compensator design [7].

On the other hand, it has been well known that the discrete realization for the plant possessing multiple bending modes might cause the system degradation associated with the selection of the sampling interval. [6], [8]. Such phenomenon can be readily avoided by measuring the level of robustness against nonlinear perturbations at the selected sampling interval. This is because the robustness level get drastically reduced where there the co-relation exists between the sampling interval and the higher frequency bending modes. Hence, the proposed approach not only provides the robustness bound, not also gives us a guideline to choose a proper sampling rate.

II. ROBUSTNESS OF THE LQ STATE FEEDBACK CONTROL

In practice, the actual system is nonlinear and often subjected to parameter and structural variations, thereby it is usually difficult to acquire accurate mathematical models. It is therefore, necessary to measure the robustness of the linear-quadratic controller in the presence of nonlinear perturbations [8]. In this section, the allowable bound for nonlinear perturbations is sought, in order to deal with the control spillover resulting from modelling errors. This bound helps to quantify the effects of unmodelled residuals on the closed-loop system.

The nonlinear perturbations associated with parameter variations and modelling errors are taken into account by the addition of a vector g, namely,

$$\dot{x}(t) = Ax(t) + bu(t) + g(x(t), u(t)) \tag{1}$$

Since the exact expression of the nonlinear perturbations is not available, the control input is assumed to be generated based on the linear model; i.e., first two terms in the right-hand-side of eq.(1). Under the state feedback law $u(t) = -k^T x(t)$, the resulting closed-loop system is given by

$$\dot{x} = A_c x + g(x) \tag{2}$$

$$J = \int_0^\infty x^T(t)(Q + \frac{1}{r} k\, k^T)x(t)dt \tag{3}$$

where

$$A_c \equiv A - b\, k^T \tag{4}$$

To minimize J, the stabilizing gain k must satisfy the necessary condition.

$$k = \frac{1}{r} Sb \tag{5}$$

where S satisfies the matrix Riccati equation

$$O = A^{T}S + SA - \frac{1}{r} S\, bb^{T}S + Q \tag{6}$$

Then there exists a sufficient condition for an allowable level of nonlinear perturbation such that the stability of the closed-loop system is not disturbed. We now recall

Theorem 1. [8] Let $D = Q + \frac{1}{r} Sbb^{T}S$. The closed-loop system, given in eq.(2), remains asymptotically stable if the nonlinear vector function g satisfies the following condition:

$$\frac{\|g(x)\|}{\|x\|} < \zeta = \frac{1}{2\|D\|_s \|S\|_s} = \frac{\sigma_{min}(D)}{2\sigma_{max}(S)} \tag{7}$$

where $\|\cdot\|$ and $\|\cdot\|S$ denote the Euclidean and the spectral norms, respectively. Moreover, $\sigma_{max}(S)$ denotes the largest singular value of S, while $\sigma_{min}(D)$ denotes the smallest singular value of D.

The bound for the nonlinear perturbation that guarantees stability of the closed-loop system can be obtained by considering a suitable Lyapunov function, namely,

$$V(x) = x^{T}Sx \tag{8}$$

where S is the solution of eq. (6). The time-rate of change is then given by

$$\dot{V}(x) = \dot{x}^{T}Sx + x^{T}S\dot{x} \tag{9}$$

Using eq.(2), the above expression becomes

$$\dot{V}(x) = x^{T}(A_c^{T}S + SA_c) + 2g^{T}Sx \tag{10}$$

$$= x^{T}(A^{T}S + SA - \frac{1}{r} Sbb^{T}S + Q)x - x^{T}(Q + \frac{1}{r} Sbb^{T}S)x + 2g^{T}Sx \tag{11}$$

Since the first quadratic form becomes zero by virtue of eq.(6), we obtain the following Lyapunov equation

$$A_c^{T}S + SA_c = -D \tag{12}$$

where

$$D = Q + \frac{1}{r} Sbb^{T}S \tag{13}$$

It should be noted that D is positive-definite due to the stabilizing characteristic of the LQ state feedback. Moreover, S is symmetric and positive-definite when D is symmetric and positive-definite, provided that Ac is asymptotically stable. Then, the time derivative of eq.(8) becomes

$$\dot{V}(x) = -x^T Dx + 2g^T Sx \tag{14}$$

If the closed-loop system is stable, $\dot{V}$ must be negative definite, which requires that

$$\dot{V} = -x^T Dx + 2g^T Sx < 0, \quad \forall x \neq 0 \tag{15}$$

For the preceding inequality to hold, the following condition has to be met,

$$\max_{g, x \in \mathbb{R}^{2n}} |g^T Sx| < \frac{1}{2} \min_{x \in \mathbb{R}^{2n}} (x^T Dx) \tag{16}$$

The left hand side of eq.(16) leads to

$$\max_{g, x \in \mathbb{R}^{2n}} | g^T Sx| \le ||g|| \, ||Sx|| \le ||g|| \, ||S||_s ||x|| = \sigma_{\max}(S) ||g|| \, ||x|| \tag{17}$$

where $||\cdot||_s$ denotes the matrix spectral norm, defined as

$$||S||_s = \sigma_{\max}(S) \tag{18}$$

and $\sigma_{\max}(S)$ is the largest singular value of S. Consequently, eq.(17) becomes

$$\max_{g, x \in \mathbb{R}^{2n}} | g^T Sx| \le \sigma_{\max}(S) ||g|| \, ||x|| \tag{19}$$

Moreover, the right-hand-side of eq.(16) satisfies

$$\max_{x \in \mathbb{R}^{2n}} | x^T Dx | \ge \sigma_{\min}(D) ||x||^2 \tag{20}$$

since D is symmetric positive definite [11], namely,

$$\sigma_{\min}(D) ||x||^2 \le x^T Dx \le \sigma_{\max}(D) ||x||^2 \tag{21}$$

where $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$ denote the smallest and the largest singular values of D, respectively. Summing eqs.(17) and (20) leads to the bound for the nonlinear perturbation function g(x) such that the closed-loop system of eq.(2) remains stable in the presence of nonlinear perturbations, namely,

$$\frac{||g(x)||}{||x||} < \zeta \equiv \frac{\sigma_{\min} D}{2\sigma_{\max}(S)} = \frac{1}{2||D^{-1}||_s ||S||_s} \tag{22}$$

Ⅲ. ROBUSTNESS OF THE DIGITAL CONTROL LAWS

Ⅲ. Ⅰ. The Shift Operator

The discrete-time system representation using the shift form can be written as

$$x(k+1)=A_q x(k)+b_q u(k)+g(x(k),u(k)) \tag{23}$$

where the vector $g(x(k),u(k))$ denotes the nonlinear perturbations associated with the discrete-time realizations due to finite-word-length effects such as roundoff errors. The control input is then assumed to be generated by the linear model

$$x(k+1)=A_q x(k)+b_q u(k) \tag{24}$$

such that

$$u(k)=-k_q^T x(k) \tag{25}$$

Here, kq is the steady-state, discrete-time controller gain, which minimizes the continuous-time cost function given by

$$J=\frac{1}{2}\int_{k=0}^{Nh}(x^T(t)Q_c x(t)+q_c u^2(t))d\tau \tag{26}$$

in which Qc is symmetric and positive-semidefinite, while q_c is a positive real number. In an effort to attain the performance of the LQ state feedback in continuous-time, the discrete equivalent of the continuous cost function is used, namely

$$J=\frac{1}{2}\sum_{k=0}^{N-1}[x(k)\ u(k)]^T\begin{bmatrix} Q_{11} & q_{12} \\ q_{12}^T & \zeta_q \end{bmatrix}\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{27}$$

where

$$\begin{bmatrix} Q_{11} & q_{12} \\ q_{12}^T & \zeta_q \end{bmatrix}=\int_0^T\begin{bmatrix} A_q^T & 0 \\ b_q^T & 1 \end{bmatrix}\begin{bmatrix} Q_c & 0 \\ 0^T & q_c \end{bmatrix}\begin{bmatrix} A_q & b_q \\ 0^T & 1 \end{bmatrix}d\tau \tag{28}$$

An algorithm for obtaining the solution of eq.(28) is well established and is readily available in commercial software packages. It should be noted that the resulting discrete weighting matrices include cross terms, containing the product of x and u. Such cross terms can be eliminated by defining a fictitious control input such that

$$u_f = \sigma^T x + u \tag{29}$$

where $\sigma = \frac{1}{\zeta_q} q_{12}$,Then, eq.(27) becomes

$$J = \frac{1}{2} \sum_{k=0}^{N-1} [x(k)\ u_f(k)]^T \begin{bmatrix} \Psi_q & 0 \\ 0^T & \zeta_q \end{bmatrix} \begin{bmatrix} x(k) \\ u_f(k) \end{bmatrix} \tag{30}$$

where

$$\Psi_q = Q_{11} - q_{21}\sigma^T \tag{31}$$

The gain is then obtained by virtue of eq.(29), namely,

$$k_q = k_f + \sigma \tag{32}$$

where

$$k_f = (\zeta_q + b_q^T S_q b_q)^{-1} A_q^T S_q^T b_q \tag{33}$$

Sq satisfies the following discrete algebraic Riccati equation

$$O = S_q - A_q^T S_q A_q + (\zeta_q + b_q^T S_q b_q)^{-1} A_q^T S b_q b_q^T S_q^T A_q - \Psi_q \tag{34}$$

The closed-loop system is thus obtained by

$$x(k+1) = \Phi_q x(k) + g(x(k)) \tag{35}$$

where

$$\Phi_q = A_q - b_q k_q^T \tag{36}$$

In eq.(35), g becomes a function of only x after applying the admissible control law. The following theorem provides a sufficient condition on the nonlinear vector function g such that the resulting closed-loop system remains stable.

Theorem 2. Let Dq be the solution of the following discrete time algebraic Lyapunov equation :

$$\Phi_q^T S_q \Phi_q - S_q = - D_q \tag{37}$$

The discrete time, closed-loop system given in eq.(35) remains asymptotically stable if the nonlinear vector function g satisfies

$$\frac{\|g(x)\|}{\|x\|} < \zeta_q = -1 + \sqrt{1+k}, \ \forall\ x \in \mathbb{R}^{2n} \tag{38}$$

where

$$k = \frac{1}{2\,\|D_q\|_s\ \|S_q\|_s} \equiv \frac{\sigma_{min}(D_q)}{2\sigma_{max}(S_q)} > 0 \tag{39}$$

in which $\|\cdot\|$ and $\|\cdot\|_s$ denote the Euclidean and spectral norms, respectively. Moreover, σ_{max}(Sq) is the largest singular value of Sq, while σ_{min}(Dq) is the smallest singular value of Dq. The theorem can be proved by defining

$$V(x(k)) = x^T S_q x \tag{40}$$

where Sq is the solution of eq.(34). Taking the difference of the foregoing Lyapunov function produces

$$\triangle V = V(x(k+1)) - V(x(k)) \tag{41}$$

Substitution of eq.(23) into eq.(41) leads to

$$\begin{aligned}\triangle V &= x^T(\Phi_q^T S_q \Phi_q - S_q)x^T + 2g^T \Phi_q x + g^T S_q g_q \\ &= - x^T D_q x + 2 g^T S_q \Phi_q x + g^T S_q g\end{aligned} \tag{42}$$

where

$$\Phi_q^T S_q \Phi_q - S_q = - D_q \tag{43}$$

Moreover, Dq is positive-definite due to the stabilizing characteristic of the LQ state feedback. It should be realized that eq.(43), for the given positive-definite Dq, has a unique solution for Sq and this Sq is positive definite, provided that the closed-loop system is asymptotically stable. The stability condition of the closed-loop system requires

$$\triangle V = - x^T D_q x + 2 g^T S_q \Phi_q x + g^T S_q g < 0, (\forall x \neq 0) \tag{44}$$

To satisfy the foregoing inequality, the following condition has to be met :

$$\max_{x \in \mathbb{R}^{2n}} |2 g^T S_q \Phi_q x + g^T S_q g| < \min_{x \in \mathbb{R}^{2n}} (x^T D_q x) \tag{45}$$

From the left-hand side of eq.(45),

$$\begin{aligned}&\max_{g, x \in \mathbb{R}^{2n}} |2 g^T S_q \Phi_q x + g^T S_q g| \\ &\le 2\| g^T\| \, \| S_q\| \, \| \Phi_q x\| + g^T S_q g \\ &\le 2\|g^T\| \, \|S_q\|_s \, \| \Phi_q\|_s \|x\| + \sigma_{\max} (S_q) \|g\|^2 \\ &\le 2\sigma_{\max}(S_q)\| g^T\| \, \|x\| + \sigma_{\max} (S_q) \|g\|^2\end{aligned} \tag{46}$$

The preceding derivation requires that all the eigenvalues of Φ_q have a magnitude less than 1. This is true if the closed-loop system given in eq.(35) is stable. Hence, the largest eigenvalue of Φ_q, which is also the spectral norm of $\| \Phi_q\|_s$, has magnitude less than 1. In light of eq.(21), the right hand side of eq.(45) must hold, namely

$$\min_{x \in \mathbb{R}^{2n}} (x^T D_q x) \ge \sigma_{\min} (D_q) \|x\|^2 \tag{47}$$

Assembling eq.(46) and eq.(47) leads to

$$\sigma_{\max} (D_q) \|g\|^2 + 2 \sigma_{\max} (S_q)\| g^T\| \|x\| - \sigma_{\min} (S_q) \|x\|^2 < 0 \tag{48}$$

Since $\| \cdot \|$ is always greater than zero by definition, there is only one solution that satisfies eq.(48), namely,

$$\frac{\|g(x)\|}{\|x\|} < -1+\sqrt{1+k} \tag{49}$$

where

$$k \equiv \frac{\sigma_{\min}(D_q)}{2\,\sigma_{\max}(S_q)} \tag{50}$$

Ⅲ.Ⅱ. The Euler Operator

The discrete-time system representation using the Euler operator is defined as

$$\varepsilon \equiv \frac{z-1}{T} \tag{51}$$

where z is the shift operator and T the sampling interval. When expressed in terms of the Euler operator, the discrete-time, closed-loop system given in eq.(35) has the form

$$\varepsilon x(k) = \Phi_\varepsilon x(k) + g(x(k)) \tag{52}$$

where

$$\Phi_\varepsilon = A_\varepsilon - b_\varepsilon k_\varepsilon^T \tag{53}$$

The desired gain is then obtained by

$$k_\varepsilon = k_f + \sigma \tag{54}$$

where $\sigma = \frac{1}{\zeta_q} q_{12}$. To use the same cost function as obtained in the shift operator formulation, the fictitious gain kf is chosen to minimize the cost function given below :

$$J = \frac{T}{2}\sum_{k=0}^{N-1}[x(k)u_f(k)]^T \begin{bmatrix} \Psi_\varepsilon & 0 \\ 0^T & \zeta_\varepsilon \end{bmatrix} \begin{bmatrix} x(k) \\ u_f(k) \end{bmatrix} \tag{55}$$

where

$$\Psi_\varepsilon = \frac{\Psi_q}{T} \tag{56}$$

$$\zeta_\varepsilon = \frac{\zeta_q}{T} \tag{57}$$

Finally,

$$k_\varepsilon^T = (\zeta_\varepsilon + T b_\varepsilon^T S_\varepsilon b_\varepsilon)^{-1} b_\varepsilon^T S_\varepsilon (1 + T A_\varepsilon) \tag{58}$$

where S_ε satisfies the following discrete-time Riccati equation [5]

$$\begin{aligned} O = {} & \Psi_\varepsilon + S_\varepsilon A_\varepsilon + A_\varepsilon^T S_\varepsilon + T A_\varepsilon^T S_\varepsilon A_\varepsilon \\ & - (\zeta_\varepsilon + T b_\varepsilon^T S_\varepsilon b_\varepsilon)^{-1} k_\varepsilon k_\varepsilon^T \end{aligned} \tag{59}$$

The same sufficient condition as given in Theorem 2 can also be written in the Euler form:

Theorem 3. Let D_ε be the solution of the following discrete algebraic Lyapunov equation:

$$\Phi_\varepsilon^T S_\varepsilon + S_\varepsilon \Phi_\varepsilon^T + T\Phi_\varepsilon^T S_\varepsilon \Phi_\varepsilon = -D_\varepsilon \quad (60)$$

The discrete time, closed-loop system given in eq.(52) remains asymptotically stable if the nonlinear vector function g satisfies

$$\frac{\|g(x)\|}{\|x\|} < \zeta_\varepsilon = \frac{1}{T}(-1+\sqrt{(1+Tk)}) \quad (61)$$

where

$$k = \frac{1}{2\|D_\varepsilon\|_s \|S_\varepsilon\|_s} \equiv \frac{\sigma_{\min}(D_\varepsilon)}{2\sigma_{\max}(S_\varepsilon)} > 0 \quad (62)$$

in which $\|\cdot\|$ and $\|\cdot\|_s$ denote the Euclidean and spectral norms, respectively. Moreover, $\sigma_{\max}(S_\varepsilon)$ is the maximum singular value of S_ε, while $\sigma_{\min}(D_\varepsilon)$ is the minimum singular value of D_ε. A Lyapunov function candidate is selected as

$$V = x^T(k)\ S_\varepsilon x(k) \quad (63)$$

where S_ε is the solution of eq.(59). The difference rate of the Lyapunov function is performed using the Euler operator, i.e.,

$$\varepsilon V = \varepsilon x^T\ S_\varepsilon x(k) + x^T(k)\ S_\varepsilon \varepsilon x(k) + T\,(\varepsilon x(k))^T(\varepsilon x(k)) \quad (64)$$

Upon substituting eq.(52) into the foregoing equation, we obtain

$$\varepsilon V = -x^T(D_\varepsilon)x + 2g^T S_\varepsilon(1+T\Phi_\varepsilon)x + Tg^T S_\varepsilon g \quad (65)$$

where

$$-D_\varepsilon \equiv \Phi_\varepsilon^T S_\varepsilon + S_\varepsilon \Phi_\varepsilon^T + T\Phi_\varepsilon^T S_\varepsilon \Phi_\varepsilon \quad (66)$$

Notice that D_ε is positive-definite due to the stabilizing characteristic of the LQ state feedback. Moreover, there exists a unique solution for S_ε which satisfies eq.(66). Furthermore, S_ε is positive-definite if the closed-loop system given in eq.(52) is stable [5].

For the nonlinear vector function g, the bound that keeps the closed-loop system asymptotically stable, can then be obtained from the relation given below

$$\varepsilon V = -x^T D_\varepsilon x + 2g\,S_\varepsilon(1+T\,\Phi_\varepsilon)x + T g^T S_\varepsilon g < 0 \quad (67)$$

Since $\Phi_q = 1 + T\Phi_\varepsilon$, it follows that

$$2g\,S_\varepsilon \Phi_q x + T g^T S_\varepsilon g < x^T D_\varepsilon x \quad (68)$$

In order to satisfy the preceding equality, the following relation should be satisfied

$$\max_{g,x \in \mathbb{R}^{2n}} |2g^T S_\varepsilon \Phi_q x + T g^T S_\varepsilon g| < \min_{x \in \mathbb{R}^{2n}} (x^T D_\varepsilon x) \quad (69)$$

Considering that eigenvalues of Φ_q have magnitude less than unity, the left hand side of eq.(69) satisfies

$$\max_{g,x \in R^{2n}} |2 g^T S_\varepsilon \Phi_q x + T g^T S_\varepsilon g| \le 2 \|S_\varepsilon\| \|g\| \|x\| + T g^T S_\varepsilon g \tag{70}$$

$$\le 2\sigma_{max} S_\varepsilon)\|g\| \|x\| + T\sigma_{max} (S_\varepsilon) \|g\|^2$$

Since $D\varepsilon$ is symmetric and positive-definite, the right hand side of eq.(69) satisfies

$$\min_{x \in R^{2n}} (x^T D_\varepsilon x) \ge \sigma_{min} (D_\varepsilon) \|x\|^2 \tag{71}$$

Summing eq.(71) and eq.(71), we obtain

$$\|g\|^2 + \frac{2}{T} \|x\| \|g\| - \frac{1}{T} k \|x\|^2 < 0 \tag{72}$$

where

$$k \equiv \frac{\sigma_{min}(D_\varepsilon)}{\sigma_{max}(S_\varepsilon)} > 0 \tag{73}$$

Since $\|g\| > 0$ if $g \ne 0$, the bound on $\|g\|$ that satisfies eq.(72) is obtained as

$$\frac{\|g(x)\|}{\|x\|} < \frac{1}{T}(-1 + \sqrt{(1+Tk)}) \tag{74}$$

A key aspect of the Euler operator is that all discrete-time quantities converge to the corresponding continuous-time quantities as the sampling rate increases, whereas these convergences are not obvious when using the shift operator. To prove the convergence of the discrete-time solution to the corresponding solution for the underlying continuous-time problem, the limit of the partial derivative of eq.(61) with respect to T, as the sampling interval approaches zero, is taken, namely,

$$\lim_{T \to 0} \frac{\partial}{\partial T} (1+Tk)^{1/2} = \frac{1}{2} k = \frac{\sigma_{min}(D_\varepsilon)}{2\sigma_{max}(S_\varepsilon)} \tag{75}$$

Moreover, the following relations are known to hold [5]:

$$\lim_{T \to 0} \sigma_{max}(S_\varepsilon) = \sigma_{max}(S) \tag{76}$$

$$\lim_{T \to 0} \sigma_{min}(D_\varepsilon) = \sigma_{min}(D) \tag{77}$$

Then,

$$\lim_{T \to 0} \zeta_\varepsilon = \zeta \tag{78}$$

where ζ is defined in Theorem 1. The preceding result shows that the results obtained using the Euler operator are close representations of the corresponding continuous-time results when a high sampling rate is used.

IV. NUMERICAL EXAMPLES

Taking a rotating flexible beam as an example, we apply the above theorems to assess the allowable level of nonlinear perturbations arising from the use of discrete-time LQ state feedback. In these studies, the rotating flexible beam is used as a model to be controlled. The vibrational behaviour of the beam is described using cubic splines[16].

Table 1. Material properties of the beam

number of nodal points(cubic-spline model)	5
number of modes to be considered (normal-mode model)	4
mass per unit length(m)	0.6697 [kg/m]
flexural rigidity (EI)	14.8535[kgm^3/s^2]
moment of inertia of the hub (I_h)	2.0927×10^{-4}[kg-m^2]
moment of inertia of the unflexed rigid beam (I_b)	0.2232 [kg-m^2]
length (L)	1 [m]
cross-section	0.0762×0.0032[m^2]

For the given material properties, the system matrices can be obtained as follows;

$$A = \begin{bmatrix} 0 & 1 \\ -M^{-1}K & 0 \end{bmatrix} \in \mathbb{R}^{(2n\times 2n)} \tag{79}$$

$$b=\begin{bmatrix} 0 \\ M^{-1}p \end{bmatrix} \in \mathbb{R}^{2n} \quad (80) \qquad C=\begin{bmatrix} \zeta \\ 0 \end{bmatrix} \in \mathbb{R}^{2n} \quad (81)$$

where

$$M=\begin{bmatrix} 0.4469 & 0.0244 & 0.0355 & 0.0179 & 0.0054 \\ 0.0244 & 0.0027 & 0.0039 & 0.0020 & 0.0006 \\ 0.0355 & 0.0039 & 0.0059 & 0.0031 & 0.0010 \\ 0.0179 & 0.0020 & 0.0031 & 0.0018 & 0.0006 \\ 0.0054 & 0.0006 & 0.0010 & 0.0006 & 0.0002 \end{bmatrix} \quad (82) \quad K=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1.2378 & 0.6189 & 0 & 0 \\ 0 & 0.6189 & 2.4756 & 0.6189 & 0 \\ 0 & 0 & 0.6189 & 2.4756 & 0.6189 \\ 0 & 0 & 0 & 0.6189 & 2.4756 \end{bmatrix} \quad (83)$$

$$\zeta = [\ 1.0000 \quad 0.1146 \quad 0.1875 \quad 0.1250 \quad 0.0625]^T \tag{84}$$

with O and 1 denoting the $n\times n$ zero and identity matrices, respectively. Moreover, 0 is the n-dimensional zero vector and the permutation vector p is given by

$$p = [\ 1 \quad 0_{n-1}]^T \tag{85}$$

The discrete-time LQ state feedback control law is then formulated using both the Euler and the shift operators. The tolerable bounds for nonlinear perturbations, given in Theorems 2 and 3, are evaluated in terms of the sampling interval and the penalizing factors in their weighting matrices. It should be noted that the discrete cost functions, for both the Euler and the shift operator formulations, are obtained in such a way that they are equivalent to an analog cost function in continuous time. This will provide a fair basis for comparison of the aforementioned discrete-time systems relative to the continuous-time system.

To facilitate the comparison, the robustness bound for the continuous-time LQ state feedback, given in Theorem 1, is calculated with the cost function given in eq.(3), whose weighting matrices are given below, namely,

$$Q = \begin{bmatrix} p\,1_n & O_n \\ O_n & v\,1_n \end{bmatrix}, \quad r = 1$$

where p is the weighting factor for the generalized coordinates (in this simulation, p=100) and r is for the time-rate of change of the generalized coordinates, which varies 0 to 0.001. Moreover, 1n is the n-dimensional identity matrix.

Identical conditions are used for discrete-time LQ state feedback based on both the Euler and the shift operators. The robustness bounds can then be evaluated in terms of the sampling interval T and the weighting factor v.

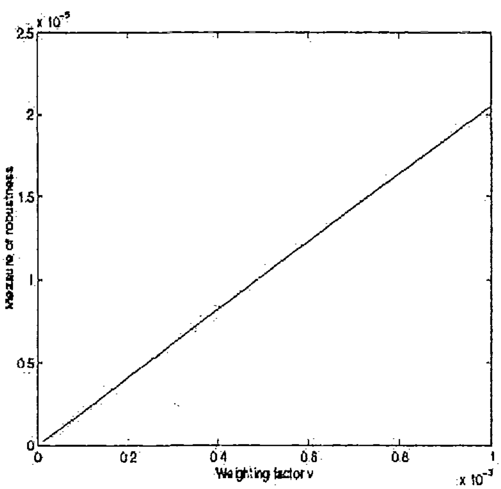

Figure1. Measure of Robustness ζ for the continuous time LQ state feedback

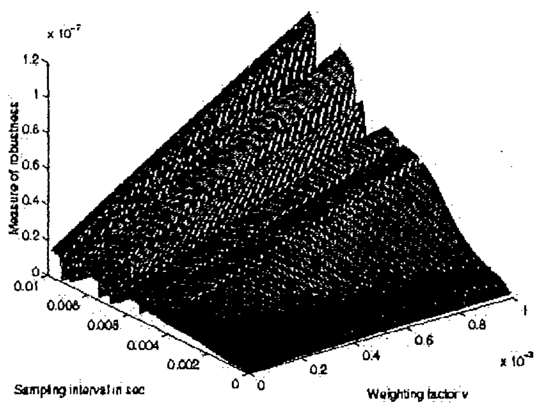

Figure2. Measure of Robustness ζ_q using the shift operator

The robustness envelop obtained using the shift operator shows that the robustness bound increases as the penalizing factor v is increased, whereas the bound decreases as the sampling rate is increased(Fig.2). In fact, when using the shift operator, the obtainable bound is reduced by as much as about 2 orders of magnitude, compared to the continuous case. Consequently, the said digital control system may become more sensitive to parameter variations and more vulnerable to disturbances as the sampling rate increases. This is contrary to the commonly made assumption that the performance of a digital controller improves as the sampling rate is increased.

On the other hand, the bound envelop obtained using the Euler operator shows that the robustness bound increases as both the penalizing factor and sampling rate are increased(Fig.3). Moreover, the overall magnitude tends to converge to that of the continuous-time case as the sampling rate increases. This suggests that a higher sampling rate allows the continuous-time system to be better approximated by the discrete-time system based on the Euler operator. It should be mentioned that unacceptable regions exist for nonlinear perturbations in both the shift and the Euler operators, and they occur at the same sampling rates (Figs. 2, 3).

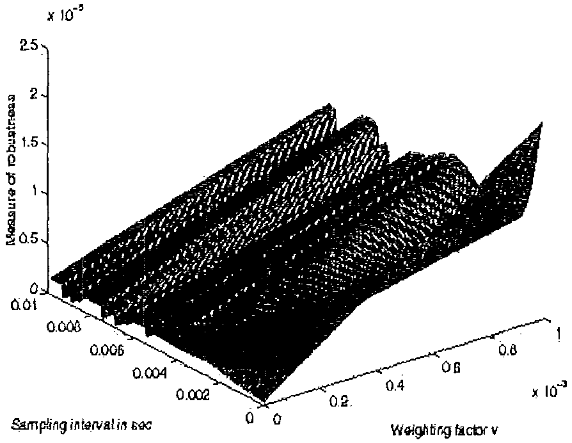

Figure3. Measure of Robustness ζ_{ε} using the Euler Operator

This implies that such regions are independent of the choice of operator, but rather dependent on the selection of the sampling interval. Such phenomenon, explained by [6], [8], is induced by the discrete-time state feedback for a plant possessing multiple bending modes with a sampling rate which is slower than twice the selected open-loop plant resonance. Under this situation, the controller has no information about the motion to be controlled, thereby producing uncontrollable system.

Considering that the plant to be controlled possesses higher-frequency bending modes, the sampling rate has to be chosen in such a way that the unacceptable regions for the nonlinear perturbations are avoided. Hence, the preceding analysis not only provides the robustness bound, but also gives us a guideline to choose a proper sampling rate.

V. CONCLUSIONS

The discrete-time realization of the control scheme has been achieved using the Euler operator, which is known to be numerically more robust than the shift operator. When using the Euler operator, the close connections between the continuous and discrete-time results were established, i.e., the discrete-time results converge to the continuous-time counterparts as the sampling rate increases. Moreover, bounds for the nonlinear perturbations were formulated in an effort to quantitatively measure the robustness of the discrete-time LQ state feedback associated with both the Euler and shift operators. Simulations for the rotating flexible beam showed that the feedback control obtained using the Euler operator is more robust against nonlinear perturbations.

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