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구매단가 및 운송비 할인을 고려한 최적 주문 정책

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〈요 약〉

본 논문은 구입량에 따라 구매가격과 운송비용이 각각 할인되는 상황하에서 주문정책을 수립할 때 총비용을 최소화시키는 적절한 주문량을 결정하는데 그 목적이 있다.

본 고에서는 구매 단가의 경우에는 단계할인이 적용되고 운송비용의 경우에는 운송단위별로 할인이 이루어지는 경우에 대한 모델이 제시되었으며 이와같은 할인 계획하에서 최적 주문량을 결정할 수 있는 절차가 제시되었다.

Optimal Order Policy with Quantity Discounts for both Purchasing Price and Freight Cost

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〈Abstract〉

This paper deals with an EOQ model in which both the unit purchasing price and the freight cost depend on the quantity of the lot size. The model adopts incremental quantity discounts for the price and a general form of discounts for the freight cost.

Investigation of the properties of an optimal solution allows us to develop an algorithm whose validity is illustrated through an example problem.

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1. INTRODUCTION

Inventory systems in which the unit variable procurement cost depends on the quantity of an order are referred to as systems with quantity discounts. The traditional quantity discount models have analyzed solely the unit purchase price discount and considered two types of price discount, "all units" and "incremental" discounts (Hadley and Whitin[1], Johnson and Montgomery [2]). Recognizing another type of discount structure, Lee[3] formulated the classical EOQ model with set up cost including a fixed cost and freight cost, where the freight cost has a quantity discount (economies of scale). Recently, Hark Hwang et al. [4] introduced an EOQ model with all unit quantity discounts for the price and general form of discounts for the freight cost.

This paper formulates other price discounts scheme instead of the model introduced by Hark Hwang et al. [4]. A relevant mathematical model is introduced in Section 2 and the properties of an optimal solution are discussed. Section 3 presents an algorithm based on the results in the previous sections. A numerical example is provided in Section 4, followed by concluding remarks in Section 5.

2. DEVELOPMENT OF THE MODEL

The assumptions of this study are essentially the same as the EOQ model except for the ordering cost and the purchasing price terms.

The following assumptions and notations are used:

1) Demand rate is constant and continuous.
2) No shortages are allowed.
3) Incremental quantity discounts are available for purchasing price.
4) The buyer pays the freight cost for the transportation of the quantity purchased where the freight cost has a quantity discount.

D ; annual demand rate

Q : an order size

M_i : i^{th} price break quantity, $i=1,2,3,\dots,m$ where $M_0<M_1<M_2<\dots<M_m$ with $M_0=0$ and M_m unlimited

P_i : price per unit for $Q\in(M_{i-1}, M_i]$ with $P_{i-1}>P_i$, $i=1,2,3,\dots,m$

$P_i(Q)$: total purchasing price of Q units,

i.e., $P_i(Q)=R_i+P_i(Q-M_{i-1})$, $i=1,2,3,\dots,m$

$R_1=0$

$R_i=\Sigma\ P_k(M_k-M_{k-1})$, $i=2,3,4,\dots,m$

N_j : j^{th} freight cost break quantity, $j=1,2,3,\dots,n$, where $N_0<N_1<N_2<\dots<N_n$ with $N_0=0$ and N_n unlimited

F_j : freight cost for $Q\in(N_{j-1}, N_j]$ where $F_{j-1}<F_j$ and $F_{j-1}/N_{j-1}>F_j/N_j$, $j=1,2,3,\dots,n$

r : inventory holding cost (percentage)

A : fixed ordering cost per each order

Note that the inequalities $F_{j-1}<F_j$ and $F_{j-1}/N_{j-1}>F_j/N_j$ are necessary to have some quantity discount in the freight cost for changing the order size from N_{j-1} to N_j.

Now, for $Q \in (N_{j-1}, N_j] \cap (M_{i-1}, M_i]$, the total ordering cost per each order, the average annual inventory holding cost and the annual purchase cost become $A+F_j$, $r(P_i(Q)/Q)Q/2$ and $(P_i(Q)/Q)D$, respectively. Let

$$TC_{ij}(Q)=(A+F_j)D/Q+r(P_i(Q)/Q)Q/2 + (P_i(Q)/Q)D. \quad (1)$$

Then the total annual variable cost, $TC(Q)$, can be expressed as

$$TC(Q)=TC_{ij}(Q) \text{ if } Q \in (M_{i-1}, M_i] \cap (N_{j-1}, N_j], \quad (2)$$
$$i=1,2,3,\ldots,m \text{ and } j=1,2,3,\ldots,n.$$

The problem is to find an ordering quantity Q^* which minimizes $TC(Q)$. Note that $0<Q<\infty$ is due to the assumptions of no shortages allowed and finite demand. $TC_{ij}(Q)$ is convex function for every i and j, and there exists a unique value Q_{ij} which minimizes each function as follows;

$$Q_{ij}=\{2D(A+F_j+R_i-P_iM_{i-1})/rP_i\}^{\frac{1}{2}} \quad (3)$$

If $Q_{ij} \in (M_{i-1}, M_i] \cap (N_{j-1}, N_j]$, we will call Q_{ij} feasible minimum point and denote by Q_{ij}^*.

To facilitate the study, we introduce Q_{i0} such that

$$Q_{i0}=\{2D(A+R_i-P_iM_{i-1})/rP_i\}^{\frac{1}{2}} \quad (4)$$

Now, we present some important observations regarding the properties of an optimal solution, with the first one coming from Lee[3] and others coming from Hark Hwang et al.[4] by some revision.

Property 1.

Suppose $P_k=P_i$ for all k, $k=1,2,\ldots,m$. And let p and q for $p<q$ be the freight discount indices such that $Q_{i0} \in (N_{p-1}, N_p]$ and $Q_{iq}=\min_j\{Q_{ij} \in (N_{j-1}, N_j]\}$, respectly. Then we need only consider $N_{p-1}, N_p, \ldots, N_{q-2}, N_{q-1}, Q_{iq}$ as candidates for the optimal solution with no purchasing price discounts.

Let $MV((a, b])$ be the minimum value of $TC(Q)$ on $Q \in (a,b]$ and $QV((a,b])$ be the ordering quantity associated with $MV((a,b])$.

Property 2.

If $Q_{i0} \in (N_{p-1}, N_p]$ for some p and $N_{p-1}>M_i$, then $MV((M_{i-1}, M_i])>MV((M_i, N_h])$ where N_h is the smallest freight discount point satisfying $N_h>M_i$.

<proof>

It is not difficult to verify the following results.

1) $TC_{ij}(Q) \geq TC_{ij}(N_j)$ for $Q \in (N_{j-1}, N_j]$, $j=1,2,3,\ldots,p-1$ (5)

2) $TC_{ij-1}(N_{j-1})>TC_{ij}(N_j)$, $j=1,2,3,\ldots,p-1$ (6)

Let $N_h \in (M_{w-1}, M_w]$ for some $w>i$ and $QA=QV((M_{i-1}, M_i])$ with $QA \in (N_{k-1}, N_k]$ for some k.

We observe $TC_{ih}(N_h)>TC_{wh}(N_h)$ due to $P_w<P_i$, $P_w(Q)<P_i(Q)$ and $TC_{ik}(QA)>TC_{ih}(N_h)$ form (5) and (6). [Q.E.D.]

Property 2 suggests that we do not

need to search the interval$(M_{i-1}, M_i]$ to find an optimal solution if $Q_{i0} \in (N_{p-1}, N_p]$ and $N_{p-1} > M_i$.

Let $S_M = \{M_1, M_2, \ldots, M_m\}$ and $S_N = \{N_1, N_2, \ldots, N_n\}$. We define a set S_U as $S_U = S_M \cup S_N = \{U_1, U_2, \ldots, U_u\}$ and number each element in increasing sequence such that $U_1 < U_2 < \ldots < U_u$. Let CP_v and CF_v be the unit purchase price and the freight cost on $I_v = (U_{v-1}, U_v]$, respectively.

Property 3.

There exists at least one feasible Q_{ij} (i.e. Q_{ij}^*).

⟨proof⟩

One of the following three relationships between I_v and I_{v+1} holds.

$CP_v = CP_{v+1}$ and $CF_v < CF_{v+1}$
$CP_v > CP_{v+1}$ and $CF_v = CF_{v+1}$
$CP_v > CP_{v+1}$ and $CF_v < CF_{v+1}$

Let $TC_v(Q)$ be the total cost computed with CP_v and CF_v. Since $TC_v(Q)$ is convex, $TC_v(Q)$ has the minimum value at

$$Q_v = \{2D(A + CF_v + R_v - CP_v U_{v-1}) / rCP_v\}^{1/2},$$
$$R_v = \Sigma\ CP_k(U_k - U_{k-1}) \qquad (7)$$

and $Q_v < Q_{v+1}$ for $v = 1, 2, \ldots, u-1$.

If there were no feasible Q_v, either $Q_v > U_v$ or $Q_v \leq U_{v-1}$ for every v. Hence, either $Q_u > U_u$ or $Q_1 \leq U_0$ holds, which contradicts the feasibility of Q, i.e., $0 < Q < \infty$. [Q.E.D.]

Property 4.

Suppose Q_{fg}^* be the largest among Q_{ij}^*. Then $MV((M_{f-1}, M_f]) < MV((M_f, M_m])$.

⟨proof⟩

Suppose $Q_{fg}^* \in (U_{v-1}, U_v]$ for some v. Then Q_i defined in equation (7) is not larger than U_{i-1} for $i > v$, due to the definition of Q_{fg}. Therefore, TC(Q) is increasing convex function on $I_i = (U_{i-1}, U_i]$, $i > v$, and has a upward jump discontinuity at U_i whenever U_i is freight discount quantity. And TC(Q) is continuous function that has decreasing slope at U_i whenever U_i is price discount quantity. Hence, the interval that is greater than M_f can not contain the optimal order quantity which minimize the total cost. [Q.E.D.]

Property 4 suggests that we do not need to search the interval $(M_f, M_m]$ to find an optimal solution if Q_{fg}^* is the largest among Q_{ij}^*.

Using the properties introduced and the incremental quantity discount are that the total cost curve is continuous and the slope is decrease at price break quantities, we propose a solution algorithm from which an optimal order quantity can be obtained.

3. SOLUTION ALGORITHM

Step 1. Compute Q_{ij} by equation (3) and find Q_{fg}^* which is the largest amount among feasible Q_{ij}.
Let $MV((M_f, M_m]) = \infty$ and $QV((M_f, M_m]) = \infty$.

Step 2. If f=0, then $Q^* = QV((M_f, M_m])$ and stop.

Otherwise, compute Q_{f0} by equation (4) and determine p satisfying $Q_{f0} \in (N_{p-1}, N_p)$.

Step 3. If $N_{p-1} > M_f$, set f=f-1 and go to step 2.
Otherwise, find the smallest index r satisfying $M_{f-1} < N_r$.

Step 4. If $N_r > N_{p-1}$, LB=r and compute Q_{fr} by equation (3) and go to step 5.a.
Otherwise, LB=p-1 and go to step 5.b.

Step 5.

a. If $Q_{fr} > M_{f-1}$, then compute Q_{ij} by equation (3) and find the smallest index q, q≥r, satisfying $Q_{fq} \le N_q$.
Otherwise, set f=f-1 and go to step 2.

b. Compute Q_{ij} by equation (3) and find the smallest index q, q>p-1, satisfying $Q_{fq} \le N_q$.

Step 6. If $Q_{fq} < M_f$, then go to step 7.a.
Otherwise, find the largest index s satisfying $N_s \le M_f$ and go to step 7.b.

Step 7.

a. If LB>q-1, let $MV((M_{f-1}, M_f])$ be the $TC_{fq}(Q_{fq}^*)$ by (2) and $QV((M_{f-1}, M_f])=Q_{fq}^*$.
Otherwise, compute total cost by (2) at $Q=Q_{fq}^*$ and N_j for j=LB, LB+1,..., q-1 and find $MV((M_{f-1}, M_f])$ and $QV((M_{f-1}, M_f])$.

b. If LB>s, set f=f-1 and go to step 2.
Otherwise, compute total cost by (2) at $Q=N_j$ for j=LB, LB+1,..., s-1, s and find $MV((M_{f-1}, M_f])$ and $QV((M_{f-1}, M_f])$.

Step 8. If $MV((M_{f-1}, M_f]) < MV((M_f, M_m])$, then $MV((M_f, M_m])=MV((M_{f-1}, M_f])$ and $QV((M_f, M_m])=QV((M_{f-1}, M_f])$ and go to step 9.
Otherwise, go to step 9.

Step 9. Set f=f-1 and go to step 2.

4. NUMERICAL EXAMPLE

To illustrate the algorithm developed, we consider the following data for the problem of determining the optimal lot size.

1) A=$700, D=3000/year and r=0.2
2) Vender's price schedule

order size	unit price
0 <Q≤1500 (M_1=1500)	P_1=$20.0
0 <Q≤4000 (M_2=4000)	P_2=$19.0
0 <Q≤ (M_3=∞)	P_3=$18.5

3) Freight cost
$N_j = j * 400$, j=0, 1, 2, 3, ...,
$F_j = 400 * j * (1-0.02*(j-1))$ for j=1, 2, ...,25
$F_{25}+8*(j-25)$ for J=26, 27, ..., ∞.

An optimal order quantity can be obtained through the following steps.
<Cycle 1>

Step 1. Q_{fg}=2713.708 wher f=2 and g=7.
Let $MV((M_2, ∞])=∞$ and $QV((M_2, ∞])=∞$.

Step 2. Since Q_{20}=1863.782, p=5.

Step 3. Since $N_4=1600<M_2=4000$, the index r satisfying $M_1=1500<N_r$ becomes r=4.

Step 4. Since $N_4=N_4$, LB=4 and go to

step 5. b.

Step 5.

b. Since $Q_{27}=2713.708 < N_7=2800$, $q=7$.

Step 6. Since $Q_{27}=2713.708 < M_2=4,000$, go to step 7. a.

Step 7.

a. Since $LB=4 < q-1=6$, $MV((M_1, M_2])=\min\{TC_{27}(2713.708), TC_{24}(1600), TC_{25}(2000), TC_{26}(2400)\}=67010$ and $QV((M_1, M_2])=2,000$.

Step 8. Since $MV((M_1, M_2]) < MV((M_2, \infty])$, reset $MV((M_2, \infty])=67010$ and $QV((M_2, \infty])=2000$.

Step 9. Reset $f=f-1$ and go to step 2.

〈Cycle 2〉

Step 2. Since $Q_{10}=1024.695$, $p=3$.

Step 3. Since $N_2=800 < M_1=1500$, the index r satisfying $M_0=0 < N_r$ becomes $r=1$.

Step 4. Since $N_r=N_1 < N_{p-1}=N_2$, $LB=2$ and go to step 5. b.

Step 5.

b. Since $Q_{15}=1951.922 < N_5=2,000$, $q=5$.

Step 6. Since $Q_{15}=1951.922 > M_1=1500$, the index s satisfying $N_s < M_1=1500$ is $s=3$ and go to step 7. b.

Step 7.

b. Since $LB=2 < s=3$, $MV((M_0, M_1])=\min\{TC_{12}(800), TC_{13}(1200)\}=67030$ and $QV((M_0, M_1])=1200$.

Step 8. Since $MV((M_0, M_1]) > MV((M_1, \infty])$, go to step 9.

Step 9. Reset $f=f-1$ and go to step 2.

〈Cycle 3〉

Step 2. Since $f=0$, stop.

An optimal order quantity=2000 and the miniimum total cost=67010.

We plot the value of the objective function in Figure 1 to confirm the final results of the algorithm. For the region (a, b], TC(a) is represented by empty dot to indicate that TC(Q) is not defined on a and TC(b) by black dot in the graph.

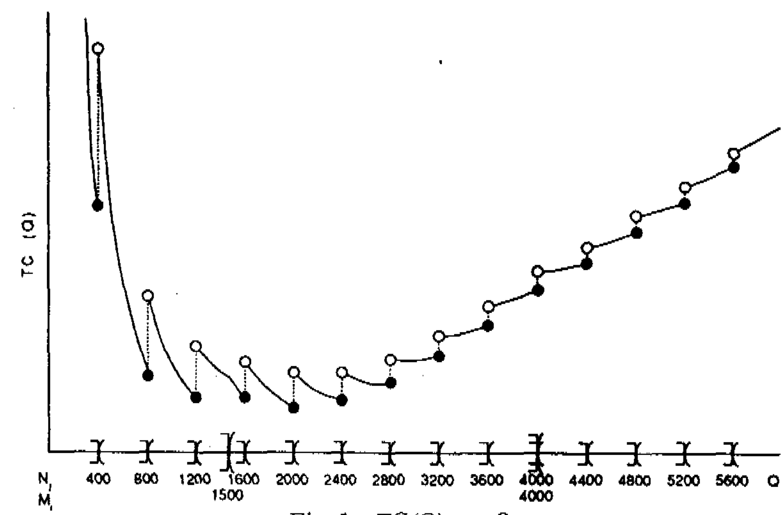

Fig. 1. *TC(Q)* vs *Q*.

5. CONCLUSION

As a marketing policy, some sellers grant discounts to customer who buy in quantify larger than that of the minimum acceptable order. As it is practiced in postal service charges, the discount is sometimes also made for the freight cost. Based on the economic order quantify inventory models, this paper analyzed how a customer can determine an optimal lot size when the opportunities for both price discount and freight cost discount available.

In this paper, the model adopted incremental quantity discounts for the price and a general form of discounts for the freight cost. Related solution algorithm was developed to solve the model and it guaranteed optimal solution.

The algorithm proposed is a little complicated in terms of the numbers of steps but bound to find an optimal solution. The algorithm can be simplified for the special scheme of the freight cost discount schedule such as Model 1 and 2 in Lee[3].

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