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A note on the Grassman manifolds $G(k, n)$

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〈Abstract〉

In this note we shall try to show that the Grassman manifold $G(k, n)$ is a differentiable manifold without using the concept of Lie group.

Grassman 다양체 $G(k, n)$에 관하여

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〈요 약〉

Lie 群의 개념없이 Grassman 다양체의 미분가능임을 보인 것이다.

As a preliminary to the definition of a differentiable manifold, we recall the definition of a topological manifold M of dimension n: it is a Hausdorff space with a countable basis of open sets and with the further property that each point has a neighborhood homeomorphic to an open subset of R^n. Each pair U, φ, where U is an open set of M and φ is a homeomorphism of U to an open subset of R^n, is called a coordinate neighborhood: to $q \in U$ we assign the n coordinates $x^1(q), \cdots\cdots x^n(q)$ of its image $\varphi(q)$ in R^n-each $x^i(q)$ is a real-valued function on U, the ith coordinate function. If q lies also in a second coordinate neighborhood V, ϕ, then it has coordinates $y^1(q), \cdots y^n(q)$ in this neighborhood. Since φ and ϕ are homeomorphisms, this defines a homemorphism

$$\phi_0\ \varphi^{-1}: \varphi(U \cap V) \to \phi(U \cap V),$$

the domain and range being the two open subsets of R^n which correspond to the points of $U \cap V$ by the two coordinate maps φ, ϕ, respectively. In coordinates, $\phi_\circ \varphi^{-1}$ is given by continuous functions

$$y^i = h^i(x^1, \cdots\cdots, x^n),\ i = 1, \cdots\cdots, n,$$

giving the y-coordinates of each $g \in U \cap V$ in terms of its x-coordinates. Similarly $\varphi_\circ \phi^{-1}$ gives the inverse mapping which expresses the x-coordinates as functions of the y-coordinates

$$x^i = g^i(y^1, \cdots\cdots, y^n),\ i = 1, \cdots\cdots, n.$$

The fact that $\varphi_0\phi^{-1}$ and $\phi_0\varphi^{-1}$ are homeomorphisms and are inverse to each other is equivalent to the continuity of $h^i(x)$ and $g^j(y)$, $i, j = 1, \cdots\cdots, n$, together with the identities

$$h^i(g^1(y), \cdots g^n(y)) \equiv y^i,\ i = 1, \cdots, n,$$

and

$$g^j(h^1(x), \cdots),\ h^n(x)) \equiv x^j,\ j = 1, \cdots n.$$

Thus every point of a topoloigical manifold M lies in a very large collection of coordinate neighborhoods, but whenever two neighborhoods overlap we have the formulas just given for change of coordinates. The basic idea that leads to differentiable manifolds is to try to select a family or subcollection of neighborhoods so that the change of coordinates is always given by differentiable functions.

Definition 1 We shall say that U, φ and V, ϕ are C^∞ compatible if $U \cap V$ nonempty implies that the functions $h^i(x)$ and $g^j(y)$ giving the change of coordinates are C^∞; this is equivalent to requiring $\varphi_0\phi^{-1}$ and $\phi_0\varphi^{-1}$ to be diffeomorphisms of the open subsets $\varphi(U \cap V)$ and $\phi(U \cap V)$ of $\boldsymbol{R}^n$.

Definition 2 A differentiable or C^∞ (or smooth) structure on a topological manifold M is a family $u = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that:

(1) the U_α cover M,

(2) for any α, β the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible,

(3) any coordinate neighborhood V, ϕ compatible with every U_α, $\varphi_\alpha \in \boldsymbol{u}$ is itself in $\boldsymbol{u}$.

A C^∞ manifold is a topological manifold together with a C^∞-differentiable structure.

In order to define the manifold $G(k, n)$, we introduce the concept of k-frame in $\boldsymbol{R}^n$.

Definition 3 A k-frame in $\boldsymbol{R}^n$ is a linearly independent set $\boldsymbol{x}$ of k elements of $\boldsymbol{R}^n$:

$$x_1 = (x_1^1,\ x_1^2, \cdots\cdots, x_1^n)$$
$$x_2 = (x_1^2,\ x_2^2, \cdots\cdots, x_2^n)$$
$$x_k = (x_k^1,\ x_2^k, \cdots\cdots, x_k^n)$$

and $F(k, n)$ is the set of all k-frams in $\boldsymbol{R}^n$.

A k-frame in $\boldsymbol{R}^n$ can be identified with the (k, n) matrix whose rows are $x_1, x_2, \cdots\cdots x_k$ which we also denote by $\boldsymbol{x}$.

It is clear that the set $\mu_{kn}(\boldsymbol{R})$ of all (k, n) real matrices is a differentiable manifold since $\mu_{kn}(\boldsymbol{R})$ can be identified with $\boldsymbol{R}^{kn}$.

We consider the function f: $\mu_{kn}(\boldsymbol{R}) \to \boldsymbol{R}$ defined by

$$f(x) = \sum_{i_1, \cdots i_k} \begin{vmatrix} x_1^{i_1} & x_1^{i_2} \cdots\cdots x_1^{i_k} \\ x_2^{i_1} & x_2^{i_2} \cdots\cdots x_2^{i_k} \\ \cdots\cdots & \\ x_k^{i_1} & x_k^{i_2} \cdots\cdots x_k^{i_k} \end{vmatrix}^2$$

Where $(i_1, i_2, \cdots\cdots, i_k)$ is a k-combination from $(1, 2, \cdots\cdots, n)$. Since f is the polinomial function of variables $x_j^i (i=1, 2\cdots, n;\ j=1, 2\cdots, k)$, f is continuous.

The inverse image $f^{-1}(0)$ is the set of all (k, n) matrices whose rank is smaller than k and it is closed in $\mu(k, n)$

Therefore $F(k, n) = \mu(k, n) - f^{-1}(0)$ is open in $\mu(k, n)$ and a differentiable manifold.

A k-frame $\boldsymbol{x} = (x_1, x_2, \cdots\cdots, x_k)$ is said to be equivalent to a k-frame $\boldsymbol{y} = (y_1, y_2, \cdots\cdots, y_k)$ iff $y_i = \sum_{j=1}^{n} \alpha_{ij} x_j$, where $\alpha = (\alpha_{ij})$ is a nonsingular (k, k) matrix, and this relation is denoted by $\boldsymbol{x} \sim \boldsymbol{y}$.

It is clear that the relation $\sim$ is an equivalence relation in $F(k, n)$.

Definition 4 Let $F(k, n)$ be the set of all k-frames and $\sim$ is the relation defined above. We define the set $G(k, n)$ to be the quotient set $F(k, n)/\sim$ with the quotient topology.

Theorem *$G(k, n)$ is a differentiable manifold with dimension $k(n-k)$*

Proof We will first show that the natural map π; $F(k, n) \to F(k, n)/\sim$ is open. Let φ_α; $F(k, n) \to F(k, n)$ defined by $\varphi_\alpha(x) = \alpha x$, where α is nonsingular (k, k) matrix, then φ_α is one-one, onto continuous and $\varphi_\alpha^{-1} = \varphi_{\alpha^{-1}}$ is continuous. So φ_α is a homeomorphism. Hence φ_α is an open mapping. Let U be an open set in $F(k, n)$, then $\pi^{-1}(\pi(U)) = \bigcup_{\alpha \in Gl(k, R)} \varphi_\alpha(U)$ is open, where $Gl(k, R)$ is the set of all nonsingular (k, k) matrices. Hence $\pi(U)$ is open, *i.e.*, π is open. Now we consider the function

$$i:\ F(k, n) \times F(k, n) \to \boldsymbol{R}$$

defined by

$$f(x, y) = \sum_{j=1}^{k} \sum_{i_1, \cdots, i_{k+1}} \begin{vmatrix} x_1^{i_1} \cdots\cdots x_1^{i_k} \\ \\ x_k^{i_1} \cdots\cdots x_k^{i_k} \\ y_{k+1}^{i_1} \cdots\cdots y_{k+1}^{i_k} \end{vmatrix}^2$$

then f is continuous. Hence $f^{-1}(0) = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \sim \boldsymbol{y}\} \subset \boldsymbol{R}$ is closed in $F(k, n) \times F(k, n)$. Therefore $G(k, n) = F(k, n)/\sim$ is Hausdorff. It remains to show that $G(k, n)$ has a covering by coordinates neighborhoods with C^∞-compatible coordinate maps.

Let $J=(j_1, \cdots\cdots, j_k)$ be an ordered subset of $(1, 2\cdots, n)$ and let J' be the remaining ordered subset of $(1, 2\cdots, n)$, that is,

$$J'=(1, 2, j_1-1, j_1+1, \cdots\cdots, j_2-1, j_2+1, \cdots\cdots, j_k-1, j_k+1, \cdots\cdots, n).$$

By x_J we denote the (k, k) submatrix

$$\begin{pmatrix} x_1^{j_1}\cdots\cdots x_1^{j_k} \\ x_2^{j_1}\cdots\cdots x_2^{j_k} \\ \\ x_k^{j_1}\cdots\cdots x_k^{j_k} \end{pmatrix}$$

and by $x_{J'}$ we denote the complementary $(k, n-k)$ submatrix obtained striking out the columns $j_1, \cdots\cdots j_k$ of x. Let $\tilde{U}_J$ be the subset of $F(k, n)$, consisting of matrices for which x_J is nonsingular, then $\tilde{U}_J=F(k, n)-g^{-1}(0)$, where g is a continuous function of $F(k, n)$ into R defined by $g(x)=|x_J|^2$, hence $\tilde{U}_J$ is open in $F(k, n)$ and let $U_J=\pi(\tilde{U}_J)$ be the corresponding open set in $G(k, n)$. Each

$$y=\begin{pmatrix} y_1^1\cdots\cdots y_1^n \\ \cdots\cdots \\ y_k^1\cdots\cdots y_k^n \end{pmatrix} \text{ in } \tilde{U}_J \text{ is equivalent to}$$

exactly one (k, n) matrix x in which x_J is the (k, k) identity matrix,

that is,

$$x=\begin{pmatrix} x_1^1\cdots \overset{j_1}{1} & \overset{j_2}{0}\cdots \overset{j_k}{0} & \cdots x_1^n \\ x_2^1\cdots 0 & 1\cdots 0 & \cdots x_2^n \\ \cdots & & \\ x_k^1\cdots 0 & 0\cdots 1 & \cdots x_k^n \end{pmatrix}$$

We define the function $\varphi_J: U_J \to u_{(k(n-k))}(R)$, identified with $R^{k(n-k)}$, by deleting the k columns corresponding to J in this representative x of $[y]$, that is $\varphi_J[y]=x_{J'}$.

If $[y]=[z]$ then there is a nonsingular (k, k) matrix such that $y=\alpha z$. Hence $y_J^{-1}y=y_J^{-1}\alpha z$. Since $y_J^{-1}\alpha=z_J^{-1}$ we have $\varphi_J(y)=\varphi_J(z)$. So φ_J is well defined. It is easy to see that φ_J is bijective and φ_J and φ_J^{-1} are continuous.

Therefore we hare an open covering $\{(U_J, \varphi_J) | J$ is the subset of k distinct elements of $\{1, 2, \cdots\cdots, n\}\}$ of $G(k, n)$ by C^∞-compatible coordinate neighborhoods.

Remark *The proof of the above theorem was quite complicated and only sketched at some points because of proving without the concept of Lie group. But because $G(k, n)$ si a Lie group, $G(k, n)$ may be shown a C^∞-manifold easily. (see*[4]*).*

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