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A Note on Weakly Singularity and Atomic Measure

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(Abstract)

Elementary properties of weakly singularity and atomic measure has been studied by R. A. Johnson and N. Y. Luther. Extending their methods in this paper, we are going to give some fundamental theorems which are set function defined on measurable set.

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〈요 약〉

약한 특이성과 아톰 측도의 기본성질은 R. A. Johnson 과 N. Y. Luther 에 의하여 연구되었다. 이 논문에서는 그들의 방법을 확장하여 가측집합 위에서 정의된 집합한수에 관한 몇개의 기본정리를 보이고자 한다.

I. Introduction

All measures in this paper are measures on a σ-ring $\mathfrak{S}$ of subsets of X. For each A in $\mathfrak{S}$, we denoted by μ_A the ***contracted set function*** defined on $\mathfrak{S}$ by the formura $\mu_A(E)=\mu(A\cap E)$; clearly μ_A is a measure on $\mathfrak{S}$.

The purpose of this paper is to investigate elementary properties of this measure. For example, as is well know, form of the Lebesgue decomposition theorem as stated in Theorem (2, 1) has been studied by R. A. Johnson and N. Y. Luther. We apply it to the study of the relations between weakly singularity and absolutely continuity in § II.

Finally, in § III elementary properties of atomic and nonatomic measures are investigated.

II. Weakly singularity and absolutely continuity.

Lex X be a set, $\mathfrak{S}$ be a σ-ring of subsets of X, and μ, ν be measures on $\mathfrak{S}$. We say that ν is ***absolutely continuous*** with respect to μ, denoted $\nu\ll\mu$, if $\nu(E)=0$ whenever $E\in\mathfrak{S}$ and $\mu(E)=0$. Recall that the sets E in $\mathfrak{S}$ are called measurable. We say that a subset A of X is ***locally measurable*** (with respect to $\mathfrak{S}$) if its intersection with every measurable set is measurable, that is, if $E\cap A\in\mathfrak{S}$ for every E in $\mathfrak{S}$. We say that ν is ***singular*** with respect to μ, denoted $\nu\perp\mu$, if there exists locally measurable A such that $\nu(E\cap A)=0=\mu(E-A)$ for each $E\in\mathfrak{S}$. Finally, we say that ν is ***weakly singular*** with respect to μ, denoted $\nu S\mu$, if given $E\in\mathfrak{S}$, there exists $F\in\mathfrak{S}$ such that

$\nu(E)=\nu(E\cap F)$ and $\mu(F)=0$.

This is shown in [4] which also contains the following easily verified properties of weakly singularity that we shall need.

1. $\nu S\nu$ if, and only if, $\nu=0$.
2. If $\nu S\mu$ and $\lambda\ll\mu$, then $\nu S\lambda$.
3. If $\nu S\mu$ and $\nu\ll\mu$, then $\nu=0$.

We shall make use of the following form of the Lebesgue Decomposition Theorem in [4, Theorem 2.1. or 6, Theorem 1.1.]

Theorem 2.1. (*Lebesgue Decomposition Theorem*) *Let μ and ν be measures on a σ-ring $\mathfrak{S}$. Then there exists a unique decomposition $\nu=\nu_1+\nu_2$ of ν into the sum of measures ν_1, ν_2 on $\mathfrak{S}$ such that $\nu_1\ll\mu$, $\nu_2 S\mu$, and $\nu_1 S\nu_2$.*

It is easy to see that although singularity is a symmetric relation, weakly singularity does not possess this property. Moreover, clearly $\nu\perp\mu$ implies that $\nu S\mu$ and $\mu S\nu$. However, the converse does not hold. But we give a condition under which weakly singularity is symmetric. This remarkable theorem has been proved by R.A. Johnson in [4]

Theorem 2.2. *If ν is σ-finite and $\nu S\mu$, then given $E\in\mathfrak{S}$, there exists $F\in\mathfrak{S}$ such that $\nu(E-F)=0$ and $\mu(F)=0$. Hence $\mu S\nu$ in this case.*

Definition 2.3. For each $A\in\mathfrak{S}$, we denote by μ_A the *contracted set function* defined on $\mathfrak{S}$ by the formula $\mu_A(E)=\mu(A\cap E)$.

Clearly μ_A is a measure on $\mathfrak{S}$, and $\mu_A\ll\mu$.

Lemma 2.4. (a) *If $\nu S\mu$, then $\nu_A S\mu$ for all $A\in\mathfrak{S}$.*

(b) *If $\nu\ll\mu$, then $\nu_A\ll\mu$ for all $A\in\mathfrak{S}$.*

Proof. Follows from definition of weakly singular and absolutely continuous.

Theorem 2.5. *$\nu S\mu$ if and only if $\nu(A)=0$ whenever $\nu_A\ll\mu$.*

Proof. Suppose $\nu S\mu$. Then $\nu_A S\mu$ for all $A\in\mathfrak{S}$ by Lemma(2.4). If $\nu_A\ll\mu$, then $\nu_A=0$ so that $\nu(A)=0$.

To prove the converse, we assume that $\nu(A)=0$ whenever $\nu_A\ll\mu$. By the Lebesgue decomposition theorem, we may write $\nu=\nu_1+\nu_2$, where $\nu_1\ll\mu$, $\nu_2 S\mu$, and $\nu_1 S\nu_2$. In order to show that $\nu S\mu$, it suffices to show that $\nu_1(E)=0$ for all E in $\mathfrak{S}$. Suppose, then, that $E\in\mathfrak{S}$. Since $\nu_1 S\nu_2$, there exists F in $\mathfrak{S}$ such that $\nu_1(E)=\nu_1(E\cap F)$ and $\nu_2(F)=0$. Necessarily, $\nu_F=(\nu_1)_F$. Since $\nu_1\ll\mu$, we have $(\nu_1)_F\ll\mu$ so that $\nu_F\ll\mu$. By hypothesis, we have $\nu(F)=0$. Then $\nu_1(E)=\nu_1(E\cap F)\leqq\nu(F)=0$, and we are done.

Theorem 2.6. *$\nu\ll\mu$ if and only if $\nu(A)=0$ whenever $\nu_A S\mu$.*

Proof. Suppose $\nu\ll\mu$. Then $\nu_A\ll\mu$ for all $A\in\mathfrak{S}$ by Lemma (2.4). If $\nu_A S\mu$, then $\nu_A=0$ so that $\nu(A)=0$.

To prove the converse, we assume that $\nu(A)=0$ whenever $\nu_A S\mu$. By the Lebesgue decomposition theorem, we may write $\nu=\nu_1+\nu_2$, where $\nu_1\ll\mu$, $\nu_2 S\mu$, and $\nu_1 S\nu_2$. In order to show that $\nu\ll\mu$, it suffice to show that $\nu_2(E)=0$ for all E in $\mathfrak{S}$. Suppose, then, that $E\in\mathfrak{S}$. Since $\nu_1\ll\mu$ and $\nu_2 S\mu$, we have $\nu_2 S\nu_1$. Therefore there exists F in $\mathfrak{S}$ such that $\nu_2(E)=\nu_2(E\cap F)$ and $\nu_1(F)=0$. Necessarily, $\nu_F=(\nu_2)_F$. Since $\nu_2 S\mu$, we have $(\nu_2)_F S\mu$ so that $\nu_F S\mu$. By hypothesis, we have $\nu(F)=0$. Then $\nu_2(E)=\nu_2(E\cap F)\leqq\nu(F)=0$, and the proof is completed.

Ⅲ. Atomic and nonatomic measure.

A set $E\in\mathfrak{S}$ will be called an ***atom*** for μ if $\mu(E)>0$ and given $F\in\mathfrak{S}$, either $\mu(E\cap F)$ or $\mu(E-F)$ is 0. We notice that if E is an atom for μ and $\mu(E\cup F)>0$, then $E\cap F$ is also an atom for μ. We shall say that μ is ***purely atomic*** or simply ***atomic*** if every measurable set of positive measure contains an atom. We shall say that μ is ***nonatomic*** if there are no atoms for μ. This means that every measurable set of positive measure can be split into two disjoint measurable sets, each having positive measure. Clearly the zero measure is the only measure which is both purely atomic and nonatomic. Here we shall prepare the following theorem which has been proved by R.A. Johnson in [5].

Theorem 3.1. *If $\nu \ll \lambda + \mu$ and $\mu S \lambda$ where ν is purely atomic and μ is nonatomic, then $\mu S \nu$. Hence, if μ is nonatomic, ν is atomic, and $\nu \ll \mu$, then $\mu S \nu$.*

Theorem 3.2. *If μ is any measure on $\mathfrak{S}$, then there exist measures μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$, where μ_1 is purely atomic and μ_2 is nonatomic, $\mu_1 S \mu_2$, and $\mu_2 S \mu_1$.*

Theorem 3.3. *If μ is atomic, ν is nonatomic, and $\nu \ll \mu$, then $\mu S \nu$.*

Using these theorems, it is easy to see that a measure is purely atomic [nonatomic] if an equivalent measure is purely atomic [resp., nonatomic].

Theorem 3.4. (a) *If μ is purely atomic, then so is μ_A for each A in $\mathfrak{S}$.*

(b) *If μ is nonatomic, then so is μ_A for each A in $\mathfrak{S}$.*

Proof. If $\mu_A(E) > 0$, then $\mu(A \cap E) > 0$. Since μ is purely atomic, there exists a μ-atom F such that $F \subset (A \cap E)$. Then, given $G \in \mathfrak{S}$, either $\mu(F \cap G) = 0$ or $\mu(F - G) = 0$. Hence, at least one of $\mu_A(E \cap F)$ or $\mu_A(E - F)$ is equal to zero, and this proves (a).

To prove (b), suppose E is an atom for μ_A. Then, given $F \in \mathfrak{S}$, either $\mu_A(E \cap F)$ or $\mu_A(E - F)$ is equal to zero. Hence $A \cap E$ is an atom for μ, and this proves the result.

Theorem 3.5. (a) *μ is purely atomic if and only if $\mu(A) = 0$ whenever μ_A is nonatomic.*

(b) *μ is nonatomic if and only if $\mu(A) = 0$ whenever μ_A is purely atomic.*

Proof. To prove (a), suppose μ is purely atomic. Then μ_A is purely atomic by (a) of Theorem (3.4). Since μ_A is nonatomic by hypothesis, we have $\mu_A = 0$ so that $\mu(A) = 0$.

To prove the converse, we assume that $\mu(A) = 0$ whenever μ_A is nonatomic. In view of Theorem (3.2), we may write $\mu = \mu_1 + \mu_2$, where μ_1 is purely atomic, μ_2 is nonatomic, $\mu_1 S \mu_2$ and $\mu_2 S \mu_1$. In order to show that μ is purely atomic, it suffices to show that $\mu_2(E) = 0$ for all E in $\mathfrak{S}$. Suppose, then, that E in $\mathfrak{S}$. Since $\mu_2 S \mu_1$, there exists F in $\mathfrak{S}$ such that $\mu_2(E) = \mu_2(E \cap F)$ and $\mu_1(F) = 0$. Necessarily, $\mu_F = (\mu_2)_F$. Since $(\mu_2)_F$ is nonatomic by (b) of Theorem (3.4), we have μ_F is nonatomic so that $\mu(F) = 0$ by hypothesis. Then $\mu_2(E) = \mu_2(E \cap F) \leqq \mu(F) = 0$. and this proves (a).

To prove (b), suppose μ is nonatomic. Then μ_A is nonatomic by (b) of Theorem (3.4). Since μ_A is purely atomic by hypothesis, we have $\mu_A = 0$ so that $\mu(A) = 0$.

To prove the converse, we assume that $\mu(A) = 0$ whenever μ_A is purely atomic. In view of Theorem (3.2), we may write $\mu = \mu_1 + \mu_2$, where μ_1 is purely atomic, μ_2 is nonatomic, $\mu_1 S \mu_2$, and $\mu_2 S \mu_1$.

It suffices to show that $\mu_1(E) = 0$ for all E in $\mathfrak{S}$. Suppose, then, that E in $\mathfrak{S}$. Since $\mu_1 S \mu_2$, there exists F in $\mathfrak{S}$ such that $\mu_1(E) = \mu_1(E \cap F)$ and $\mu_2(F) = 0$. Necessarily, $\mu_F = (\mu_1)_F$. Since $(\mu_1)_F$ is purely atomic by (a) of Theorem (3.4), we have μ_F is purely atomic so that $\mu(F) = 0$ by hypothesis. Then $\mu_1(E) = \mu_1(E \cap F) \leqq \mu(F) = 0$, and this proves the result.

Corollary 3.6. (Cf. [5, Theorem 2.4]). *Suppose $\nu \ll \mu$, where μ is σ-finite.*

(a) *If μ is purely atomic, then ν is purely atomic.*

(b) *If μ is nonatomic, then ν is nonatomic.*

Proof. To prove(a), suppose ν_A is nonatomic. In order to show that ν is purely atomic, it suffices to show that $\nu(A) = 0$. Since $\nu_A \ll \mu$ by (b) of Lemma(2.4) and since μ is purely atomic, we have $\mu S \nu_A$ by Theorem (3.3). Since μ is σ-finite, we have $\nu_A S \mu$ by Theorem (2.2). Therefore we have $\nu_A = 0$ so that $\nu(A) = 0$.

To prove (b), suppose ν_A is purely atomic. It suffices to show that $\nu(A) = 0$. Since $\nu_A \ll \mu$ and since μ is nonatomic, we have $\mu S \nu_A$ by Theorem (3.1). Since μ is σ-finite, we have $\nu_A S \mu$. Therefore we have $\nu_A = 0$ so that $\nu(A) = 0$, and this proves the result.

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