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p-continuous functions

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〈Abstract〉

The concept of c-continuity is due to Gentry and Hoyle [3] and that of H-continuity owes to Long and Hamlett [6]. It is known that continuity $\Rightarrow$ H-continuity $\Rightarrow$ c-continuity and that implications are not reversible. In this paper the authors investigate and study a new function which lies between continuous function and c-continuous function but independent of H-continuous function.

p-연속 함수에 관하여

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〈요 약〉

논문 [3]과 [6]에서, "연속함수⇒H-연속함수⇒c-연속함수"가 성립되나 그 역은 성립하지 않음이 보여졌다.
이 논문에서는 연속함수와 c-연속함수 사이에, 그러나 H-연속함수와는 독립적인, 새로운 함수 p-연속함수에 관한 연구이다.

Ⅰ. Preliminary

The concept of c-continuity as a generalization of continuity is due to Gentry and Hoyle [3]. A function $f: X \longrightarrow Y$ is said to be c-continuous if for each $x \in X$ and each open set V containing $f(x)$ and having compact complement there exists an open set U containing x such that $f(U) \subset V$ [3]. Long and Hamlett [6] defined H-continuity on replacing compact complement by H-closed complement in the definition of c-continuity. We recall that a set P in a space $(X, \mathscr{T})$ is said to be H-closed if and only if for each $\mathscr{T}$-open covering $\{U_\alpha\}$ of P it has a finite subfamily $\{U_{\alpha_i}\}$ such that $P \subset \bigcup_{i=1}^{n} \{U_{\alpha_i}\}$ [6]. Continuity $\Rightarrow$ H-continuity $\Rightarrow$ c-continuity and these implications are not reversible. In this paper the authors investigate and study a new function which lies between continuous function and a c-continuous function but independent of H-continuous function.

Ⅱ. Properties of p-continuous functions

Definition 1. A function $f: X \longrightarrow Y$ is said to be p-continuous if for each $x \in X$ and each open set V containing $f(x)$ and having paracom-

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pact complement, there exists an open set U containing x such that $f(U)\subset V$. (We recall that a space X is paracompact if and only if every open cover has a locally finite open refinement.)

Proposition 1. Let $f: X \longrightarrow Y$ be a function from a topological space X into a topological space Y. Then the following statements are equivalent:

(a) f is p-continuous.

(b) If V is an open subset of Y with paracompact complement, then $f^{-1}(V)$ is an open subset of X.

(c) If K is a closed paracompact subset of Y, then $f^{-1}(K)$ is closed.

Proof. (a)$\Rightarrow$(b). If V is an open subset of Y with paracompact complement, then for each $x\in f^{-1}(V)$, V is neighbourhood of $f(x)$. Hence there is a neighbourhood U of x such that $f(U)\subset V$. Thus $f^{-1}(V)$, being a neighbourhood of each of its points, is open.

(b)$\Rightarrow$(a). Let $x\in X$ and V be an open subset of Y containing $f(x)$ and having paracompact complement. Then $f^{-1}(V)$ is an open set containing x and $f(f^{-1}(V))\subset V$.

(b)$\Rightarrow$(c). Let $K\subset Y$ be a closed, paracompact set. Then $Y-K$ is an open set with paracompact complement and, therefore, $f^{-1}(Y-K)=X-f^{-1}(K)$ is open. So $f^{-1}(K)$ is closed in X.

(c) $\Rightarrow$(b). Let $V\subset Y$ be an open set with paracompact complement. Then $Y-V$ is a closed, paracompact set and therefore $f^{-1}(Y-V)=X-f^{-1}(V)$ is closed. So $f^{-1}(V)$ is open in X.

Proposition 2. Let $f: X \longrightarrow Y$ be a p-continuous closed function from a normal space X onto a space Y. If either of the spaces X and Y is T_1, then Y is Hausdorff.

Proof. (Case Ⅰ). If the space Y is T_1, let y_1, y_2 be any two distinct points in Y, then $\{y_1\}$ and $\{y_2\}$ are closed, paracompact subsets of Y. So by proposition 1, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X. By normality of X, there exist disjoint open sets U_1 and U_2 containing $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively. Since f is closed, the sets $V_1=Y-f(X-U_1)$ and $V_2=Y-f(X-U_2)$ are open in Y. It is easily verified that V_1 and V_2 are disjoint and contain y_1 and y_2 respectively.

(Case Ⅱ). If the space X is T_1, let $f(x)=Y$. Since $\{x\}$ is closed in X, $\{f(x)\}$ is a closed subset of Y. So Y is T_1. The rest follows from case Ⅰ.

Lemma 1 [2]. The product of a paracompact space and a compact space is paracompact.

Definition 2. Let $f: X \longrightarrow Y$ be any function. Then the function $g: X \longrightarrow X\times Y$, defined by $g(x)=(x, f(x))$ is called the graph function with respect to f.

Proposition 3. If $f: X \longrightarrow Y$ is a function from a compact space X into a space Y such that the graph function g is p-continuous, then f is p-continuous.

Proof. Let $x\in X$ and V be an open set containing $f(x)$ such that $Y-V$ is paracompact. Let $p_Y: (X\times Y) \longrightarrow Y$ be a projection. Since projections are continuous, $p_Y^{-1}(V)$ is open in $X\times Y$. Again, X is compact and $Y-V$ is paracompact, therefore, by Lemma 1, $X\times(Y-V)=(X\times Y)-p_Y^{-1}(V)$ is paracompact. Thus $p_Y^{-1}(V)$ is an open set in $X\times Y$ having paracompact complement. Therefore, there exists an opan set U containing x such that $g(U)\subset p_Y^{-1}(V)$. It follows that $p_Y(g(U))=f(U)\subset V$, so that f is p-continuous.

Proposition 4. Let $f: X \longrightarrow Y$ be any function, $\wedge$ and $\wedge'$ are index sets. Then the following statements are true.

(a) If f is p-continuous and $A\subset X$, then $f/A: A \longrightarrow Y$ is p-continuous.

(b) If $\{U_\alpha : \alpha\in\wedge\}$ is an open cover of X and if for each α, $f_\alpha=f/U_\alpha$ is p-continuous, then f is p-continuous.

(c) If $\{F_\beta : \beta\in\wedge'\}$ is a locally finite closed cover of X and if for each β, $f_\beta=f/F_\beta$ is p-continuous, then f is p-continuous.

Proof: (a). Let U be an open subset of Y

with paracompact complement. Then $f^{-1}(U)$ is open and hence $(f/A)^{-1}(U)=f^{-1}(U)\cap A$ is open subset of A.

(b) Let U be an open subset of Y with paracompact complement. Then $f^{-1}(U)=\cup\{f_\alpha^{-1}(U):\alpha\in\wedge\}$ and since each f_α is p-continuous, each $f_\alpha^{-1}(U)$ is open in X and so $f^{-1}(U)$ is open in X.

(c) Let F be a closed and paracompact subset of Y. Then $f^{-1}(F)=\cup\{f_\beta^{-1}(F):\beta\in\wedge'\}$, since each f_β is p-continuous, each $f_\beta^{-1}(F)$ is closed in F_β and hence in X. Again, since $\{F_\beta:\beta\in\wedge'\}$ is locally finite closed cover of X, the collection $\{f_\beta^{-1}(F):\beta\in\wedge'\}$ is a locally finite collection of closed sets. Thus $f^{-1}(F)$ being the union of a locally finite collection of closed sets is closed [4].

Proposition 5. Let $f:X\longrightarrow Y$ be p-continuous and $A\subset X$ be such that $f(A)$ is closed in Y, then $f/A:A\longrightarrow f(A)$ is p-continuous.

Proof. Let F be closed and paracompact in $f(A)$. Since $f(A)$ is closed in Y, F is closed in Y. By proposition 1, $f^{-1}(F)$ is closed in X and hence $(f/A)^{-1}(F)=f^{-1}(F)\cap A$ is closed in A.

Proposition 6. Let $f:X\longrightarrow Y$ be continuous and $g:Y\longrightarrow Z$ be a p-continuous, then gof: $X\longrightarrow Z$ is p-continuous.

Proof. Let K be a closed and paracompact subset of Z. Then $g^{-1}(K)$ is closed and since f is continuous, $(gof)^{-1}(K)=f^{-1}(g^{-1}(K))$ is closed.

Proposition 7. Let $f:X\longrightarrow Y$ be a quotient map [4]. Then a function $g:Y\longrightarrow Z$ is p-continuous if and only if gof is p-continuous.

Proof: Necessity follows from proposition 6. To prove sufficiency, let U be an open subset of Z with paracompact complement. Then $(gof)^{-1}(U)=f^{-1}(g^{-1}(U))$ is open in Y. Since f is a quotient map, $g^{-1}(U)$ is open in Y. So g is p-continuous.

III. Comparision of p-continuity

We begin this section with producing two examples which suffice to show the relationship of p-continuous function, c-continuous, H-continuous and continuous functions.

Example 1. Let E be an infinite set and f $:(E,\mathcal{T})\longrightarrow(E,\mathcal{D})$ be the identity function with $\mathcal{T}$, the cofinite topology, and $\mathcal{D}$, the discrete topology. Then f is c-continuous and H-continuous but not p-continuous.

Example 2. Let R be the set of reals and $\mathcal{T}$ usual topology on R. Let $\mathcal{T}^*$ be the right ray topology on R. That is, $\mathcal{T}^*=\{\emptyset, R, \{(r,+\infty):r\in R)\}$. Now define $f:(R,\mathcal{T})\longrightarrow(R,\mathcal{T}^*)$ by $f(x)=x$ if $x\neq 0$ and $f(x)=1$ if $x=0$. Let $(-\infty,r)=Y$ be a closed subset of $(R,\mathcal{T}^*)$ and $Q=\{(x,r):x\leq r\}$ be a $\mathcal{T}_Y^*$-open covering of Y. Then Q is not locally finite at the point r. Since no proper closed subset of $(R,\mathcal{T}^*)$ is paracompact, therefore, by proposition 1, f is p-continuous. But this function is neither H-continuous nor continuous [6, example 4].

One may note that a continuous function is p-continuous and a p-continuous function is c-continuous. However, Examples 1 and 2 show that the converse in the both cases may not be true, and further that p-continuity is independent of H-continuity.

Proposition 8. Let $f:X\longrightarrow Y$ be p-continuous function and Y be paracompact. Then f is continuous.

Proof. Let K be closed subset of Y. Since closed subspace of paracompact space is paracompact [2], so K is closed and paracompact. And hence by proposition 1, $f^{-1}(K)$ is closed in X.

Proposition 9. Let f be a p-continuous function from X into Y. If $f(X)$ is a subset of some closed paracompact subset of Y, then f is continuous.

Proof. Let D be a closed paracompact subset of Y containing $f(X)$ and let U be any open subset of Y. Since D is closed, $Y-D$ is open. Thus $U\cup(Y-D)$ is open. But complement of $U\cup(Y-D)$ is a closed subset of D and thus it is paracompact. Now by proposition 1, $f^{-1}(U\cup(Y-D))$ is open and equals to $f^{-1}(U)$. Hence f is continuous.

Definition 3[1]. A subset M of a topological space $(X, \mathscr{T})$ is α-paracompact if every open cover by member of $\mathscr{T}$ has an open locally finite refinement by member of $\mathscr{T}$.

Lemma 2[4]. Let $\{A : A\in Q\}$ be a locally finite family of subsets, then $cl[\cup\{A : A\in Q\}]=\cup\{cl\ A : A\in Q\}$.

Definition 4[5]. A function $f : X \longrightarrow Y$ is said to be weakly continuous if for each $x\in X$ and each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U)\subset cl\ V$. (cl denotes the closure operator).

Remark 1. Weakly continuous functions may not imply p-continuous functions. For,

Example 3. Let $X=\{a,b,c\}=Y$, $\mathscr{T}=\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $Q=\{\emptyset, Y, \{b,c\}\}$. Let $f : (X,\mathscr{T}) \longrightarrow (Y, Q)$ be identity function then f is weakly continuous but not p-continuous.

Proposition 10. Let f be a weakly continuous function on X into a Hausdorff space Y. Then f is p-continuous.

Proof. Let $x\in X$ and N be an open neighbourhood of $f(x)$ having paracompact complement. Let $Y-N=M$. Now corresponding to each $m\in M$, there exist disjoint open sets $U_m(f(x))$ and $V(m)$ containing $f(x)$ and m respectively, therefore $U_m(f(x))\cap cl\ V(m)=\emptyset$. By construction, $M\subset\bigcup_{m\in M}\{V(m)\}$. Since M is paracompact it is α-paracompact [1]. By definition of α-paracompact, $\{V(m)\}$ has a locally finite $\mathscr{T}$-open ($\mathscr{T}$ is a topology defined in Y) refinement $\{V^*(m) : m\in M\}$. Clearly $U_m(f(x))\cap cl\ V^*(m)=\emptyset$ for each $m\in M$. By Lemma 2, $cl\bigcup_{m\in M}V^*(M)=\cup\{clV^*(m) : m\in M\}$. Now $f(x)\notin cl\cup\{V^*(m)\}$. Let $Y-cl\bigcup_{m\in M}\{V^*(m)=U^*$. Then $f(x)\in U^*\subset cl\ U^*\subset Y-\bigcup_{m\in M}\{V^*(m)\}\subset Y-M$. By definition of weakly continuity there exists an open set O containing x such that $f(O)\subset cl\ U^*\subset Y-M=N$. Hence f is p-continuous.

Lemma 3[4]. A metacompact T_1-space is countably compact if and only if it is compact.

Proposition 11. Let $f : X \longrightarrow Y$ be a c-continuous function. If Y is T_1 and countably compact then f is p-continuous.

Proof. Let $x\in X$ and V be an open subset of Y containing $f(x)$ having paracompact complement. Since $Y-V$ is paracompact it is metacompact[4]. And T_1 is a hereditary property, therefore, $Y-V$ is T_1. Also $Y-V$ being a closed subset of countably compact space is countably compact. Hence by Lemma 3, $Y-V$ is compact. By definition of c-continuity there exists an open set N containing x such that $f(N)\subset V$.

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