UOU Report Vol. 17. No. 1. pp. 143~146. 1986　　연구논문집 제17권 제 1 호 pp. 143~146. 1986.

On fuzzy feeble continuity

Lee, Je-Yoon · Chae, Gyu-Ihn
Dept. of Mathematics
(Received September 30, 1985)

〈Abstract〉

The aim of this paper is to introduce a new class of fuzzy continuity, named by a fuzzy feebly continuous function. We investigate the characterizations of the fuzzy feeble continuity and study the relations among weaker classes of fuzzy continuity.

Fuzzy 약 연속성에 관하여

이제윤 · 채규인
수 학 과
(1985. 9. 30. 접수)

〈요 약〉

우리는 fuzzy 연속성의 약화된 연속성을 새로이 도입하여 그 성질을 알아보고 다른 약화된 fuzzy 연속성과의 관계를 알아본다.

Ⅰ. Introduction.

Fuzzy sets were introduced into the literature by Zadeh[12] and subsequently fuzzy topological spaces and fuzzy continuity were defined and studied by C.L. Chang[2]: later studies include Nazaroff[8] and Warren[9]. Applications to control theory, aircraft configuration, strategy optimization, networks and systems, etc., may be found among various references.

Since fuzzy sets as introduced by Zadeh have the same kind of operations as a set operation in general set theories. It is, therefore, natural to extend the concept of point-set-topology to fuzzy sets, resulting in a theory of the fuzzy topology.

Weaker forms of continuity in topology have been considered by many workers using the concepts of semiopen sets, regular open set and feebly open sets (α-sets)[5,6,7]. In[1] and[6], authors introduced fuzzy semiopen sets, fuzzy regular open sets and fuzzy feebly open sets as a generalization of the open sets mentioned above in the ordinary topological spaces. And also, in[1], author defined the generalizations (in section 2) of semicontinuous functions, almost continuous functions and weakly continuous functions in fuzzy setting. We will introduce a generalization of feebly continuous functions[5] in fuzzy setting.

Ⅱ. Preliminaries.

Throughout this paper, I will denote the closed unit interval $[0,1]$ of the real line. For a set X, I^X denotes the collection of all function from X into I. A member g of I^X is called a fuzzy set in X[8]. So g is a fuzzy set in X iff $g: X \longrightarrow [0,1]$ is a function. For every $x \in X$,

$g(x)$ is called the grade of membership of x in the fuzzy set g in X. If I consists of only the points 0 and 1, then g is just the characteristic function of a subset of X, and then g is called a crisp set in X. Because fuzzy sets are real-valued functions, the notation $f \leq g$ means $f(x) \leq g(x)$ for all $x \in X$. In this case, f is said to be contained in g, or g is said to contain f. The complement g' of $g \in I^X$ is $g' = 1-g$ defined by $(1-g)(x) = 1-g(x)$, for each $x \in X$. Let $f, g \in I^X$. Then $f \vee g$ and $f \wedge g$ are fuzzy sets in X defined by $(f \vee g)(x) = \vee\{f(x), g(x)\}$ and $(f \wedge g)(x) = \wedge\{f(x), g(x)\}$, for all $x \in X$, respectively. More generally, the union $\bigcup g_\alpha$, $\alpha \in A$(an index set) (resp. the intersection $\bigcap g_\alpha$, $\alpha \in A$) of a family $\{g_\alpha : \alpha \in A, g_\alpha \in I^X\}$ is defined to be the function $\vee g_\alpha$(resp. $\wedge g_\alpha$).

Definition 2.1. Let $\phi : X \to Y$ be a function from X into Y, Let $f \in I^X$ and $g \in I^Y$. Then the inverse image of g under ϕ is the fuzzy set $\phi^{-1}(g)$ in X defined by $\phi^{-1}(g)(x) = g(\phi(x))$ for $x \in X$. The image of f under ϕ is the fuzzy set $\phi(f)$ in Y defined by $\phi(f)(y) = \vee\{f(x) : y = \phi(x)\}$ for $y \in Y$.

Lemma 2.2 Let $\phi : X \to Y$ be a function, $\{g_\alpha\}$ and $\{f_\alpha\}$ a family of fuzzy sets in Y and in X, respectively. Then:

(a) $\phi(\phi^{-1}(g_\alpha)) \leq g_\alpha$ and $\phi(\phi^{-1}(g_\alpha) \wedge f_\alpha) = g_\alpha \wedge \phi(f_\alpha)$

(b) $\phi(\wedge f_\alpha) \leqq \wedge \phi(f_\alpha)$ and $\phi(\vee f_\alpha) = \vee \phi(f_\alpha)$

(c) $\phi^{-1}(\vee g_\alpha) = \vee \phi^{-1}(g_\alpha)$ and $\phi^{-1}(\wedge g_\alpha) = \wedge \phi^{-1}(g_\alpha)$

proof. We refer to the references.

Definition 2.3. Let X be a set and $T(X)$ be a family of fuzzy sets in X. Then $T(X)$ is called a *fuzzy topology* on X[2] if it satisfies the conditions: (a) $0, 1 \in T(X)$ (b) if $f, g \in T(X)$, then $f \wedge g \in T(X)$, and (c) if $g_\alpha \in T(X)$, $\alpha \in A$, then $\vee g_\alpha \in T(X)$.

The pair $(X, T(X))$ is called a *fuzzy topological space* (abb. as *fts*). The elements of $T(X)$ are called *fuzzy open sets* in a fts X. $g \in I^X$ is *fuzzy closed* in a fts X iff $g' \in T(X)$.

Definition 2.4. For a $g \in I^X$ and a fts X, the *closure* $cl(g)$ and the *interior* $int(g)$ are defined, respectively, as $cl(g) = \wedge\{f : g \leqq f, f' \in T(X)\}$ and $int(g) = \vee\{f : f \leqq g, f \in T(X)\}$.

Lemma 2.5. Let X be a fts and $f, g \in I^X$. Then

(a) If $f \leq g$, then $cl(f) \leq cl(g)$, $int(f) \leq int(g)$

(b) $cl(cl(f)) = cl(f)$ and $int(int(f)) = int(f)$

(c) $cl(f \vee g) = cl(f) \vee cl(g)$ and $cl(f \wedge g) \leqq cl(f) \wedge cl(g)$

(d) $int(f \wedge g) = int(f) \wedge int(g)$ and $int(f) \vee int(g) \leqq int(f \vee g)$

(e) $int(1-g) = 1-cl(g)$ and $cl(1-g) = 1-int(g)$

Proof. We refer to the proof of the Theorem 2.13 in[10].

Definition 2.6. $g \in I^X$ is called a *fuzzy regular open set*(resp. a *fuzzy regular closed set*) in a fts X iff $g = int(cl(g))$ (resp. $g = cl(int(g))$).

Definition 2.7. $g \in I_X$ is called a *fuzzy semiopen set* in a fts X iff there exists an $h \in T(X)$ such that $h \leq g \leq cl(h)$. The complement of a fuzzy semiopen set is called a fuzzy semiclosed set.

Definition 2.8. Let X be a fts and $g \in I^X$. Then g is called a *fuzzy feebly open set* in X iff $g \leqq int(cl(int(g)))$. g is a *fuzzy feebly closed set* in X iff g' is fuzzy feebly open in X.

III. Fuzzy feeble continuity.

In this section, we define the fuzzy feebly continuous function and investigate its properties and relations among fuzzy continuous function, fuzzy semicontinuous function and etc. In this section, $\phi : X \to Y$ will denote a function from a fts $(X, T(X))$ into a fts $(Y, T(Y))$ and $FO(X)$, $SO(X)$ the collection of all fuzzy feebly open sets and that of all fuzzy semiopen sets in a fts X, respectively.

Definition 3.1. Let $\phi : X \to Y$ be a function. Then ϕ is said to be *fuzzy feebly continuous* if $\phi^{-1}(g) \in FO(X)$ for each $g \in T(U)$.

Remark 3.2. It was well know that $\phi : X \to Y$

is fuzzy continuous iff $\phi^{-1}(g)\in T(X)$ for each $g\in T(Y)$. Thus the fuzzy continuity implies fuzzy feeble continuity because fuzzy open sets are fuzzy feebly open sets [6, Theorem 3.3].

Definition 3.3. A function $\phi: X\to Y$ is said to be ***fuzzy semicontinuous***[1] if $\phi^{-1}(g)\in SO(X)$ for each $g\in T(Y)$.

Remark 3.4. The fuzzy feeble continuity implies fuzzy semicontinuity because fuzzy feebly open sets are fuzzy semiopen sets [6, Theorem 3.5].

Note that the converses of Remark 3.2 and 3.4 are not, in general, true, as shown by the following examples.

Example 3.5. Let $X=Y=I$ and $g_1, g_2, g_3\in I^X$ (or I^Y) defined by:

$$g_1(x)=0,\quad 0\leq x\leq\frac{1}{2}.$$
$$=2x-1,\quad \frac{1}{2}\leq x\leq 1.$$
$$g_2(x)=1.\quad 0\leq x\leq\frac{1}{4}$$
$$=-4x+2,\quad \frac{1}{4}\leq x\leq\frac{1}{2}$$
$$=0,\quad \frac{1}{2}\leq x\leq 1$$
$$g_3(x)=0,\quad 0\leq x\leq\frac{1}{4}.$$
$$=\frac{1}{3}(4x-1),\quad \frac{1}{4}\leq x\leq 1.$$

Clearly, $T(X)=\{0,\ g_1, g_2, g_3, g_1\vee g_2, 1\}$ and $T(Y)=\{0,\ g_3,\ 1\}$ are fuzzy topologies on X and Y, respectively. Let $\phi: X\to Y$ be the identity function. Then ϕ is fuzzy semicontinous but not fuzzy feebly continuous (hence, not fuzzy continuous) because $g_3\in T(Y)$, $\phi^{-1}(g_3)=g_3\in SO(X)$ but $\phi^{-1}(g_3)=g_{31}\notin FO(X)$.

Example 3.6. Let $X=Y=I$ and $g_1, g_2, g_3, g_4\in I^X$ (or I^Y) defined by:

$$g_1(x)=0,\quad 0\leq x\leq\frac{1}{2}$$
$$=2x-1,\quad \frac{1}{2}\leq x\leq 1.$$
$$g_2(x)=1,\quad 0\leq x\leq\frac{1}{4}$$
$$=-4x+2,\quad \frac{1}{4}\leq x\leq\frac{1}{2},$$
$$=0,\quad \frac{1}{2}\leq x\leq 1.$$
$$g_3(x)=0,\quad 0\leq x\leq\frac{1}{4},$$
$$=\frac{4}{3}x-\frac{1}{3},\quad \frac{1}{4}\leq x\leq 1.$$
$$g_4(x)=1,\quad 0\leq x\leq\frac{1}{4},$$
$$=-4x+2,\quad \frac{1}{4}\leq x\leq\frac{1}{2},$$
$$=4x-2,\quad \frac{1}{2}\leq x\leq\frac{3}{4},$$
$$=1,\quad \frac{3}{4}\leq x\leq 1.$$

Clearly, $T(X)=\{0,\ g_1,\ g_2,\ g_1\vee g_2,\ 1\}$ and $T(Y)=\{0,\ g_1,\ g_4,\ 1\}$ are fuzzy topologies on X and Y, respectively. Let $\phi: X\to Y$ be the identity function. Then ϕ is fuzzy feebly continuous but not fuzzy continuous because $g_4\in T(Y)$ and $\phi^{-1}(g_4)=g_4\notin T(Y)$ but $\phi^{-1}(g_4)\in FO(X)$.

In[9], author defined neighborhood (abb. as nbd) of a point x of a fts X as $n\in I^X$ such that there is $g\in T(X)$ with $g\leqq n$ and $n(x)=g(x)>0$. Similarly, in[6] authors introduced a feeble nbd of a point x of a fts X as a fuzzy set n in X such that there exists $g\in FO(X)$ such that $g\leq n$ and $g(x)=n(x)>0$.

Theorem 3.7. Let $\phi: X\to Y$ be a function. Then the following statements are equivalent:

(a) ϕ is fuzzy feebly continuous.

(b) The inverse image of every fuzzy closed set in Y is fuzzy feebly closed in X.

(c) For every $x\in X$ and every nbd n of $\phi(x)$, $\phi^{-1}(n)$ is a feeble nbd of x.

(d) For every $x\in X$ and every nbd n of $\phi(x)$, there exists a feeble nbd m of x such that $\phi(m)\leqq n$ and $m(x)=\phi^{-1}(n)(x)$.

Proof. (a)$\Longleftrightarrow$(b): it is easy to prove, because $\phi^{-1}(1-g)=1-\phi^{-1}(g)$.

(a)$\Longrightarrow$(c): since n is a nbd of $\phi(x)$, there is $g\in T(Y)$ such that $g\leq n$ and $n(\phi(x))=g(\phi(x))>0$. So $\phi^{-1}(g)\leq\phi^{-1}(n)$, $\phi^{-1}(n)(x)=\phi^{-1}(g)(x)>0$ and $\phi^{-1}(g)\in FO(X)$ because of (a). Hence $\phi^{-1}(n)$ is a feeble nbd of x.

(c)$\Longrightarrow$(d): let $m=\phi^{-1}(n)$, where n is a nbd of $\phi(x)$. Then $\phi(m)=\phi(\phi^{-1}(n))\leq n$.

(d)$\Longrightarrow$(a): let $g\in T(Y)$ and $x\in X$ such that $\phi^{-1}(g)(x)>0$. Then $g(\phi(x))>0$ and g is a nbd of $\phi(x)$. Hence, by(d) there exists a feeble nbd m of x such that $\phi(m)\leq g$ and $m(x)=\phi^{-1}(g)(x)$. Thus we have $m\leq\phi^{-1}(\phi(m))\leq\phi^{-1}(g)$. Because $g\in I^X$ is fuzzy feebly open iff for each $x\in X$ with $g(x)>0$, there exists a feeble nbd $n\leq g$ such that $n(x)=g(x)$ [6, Theorem 2.11], $\phi^{-1}(g)\in FO(X)$. Hence ϕ is fuzzy feebly continuous.

References

1. K.K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82(1981), 14—32.
2. C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24(1968), 182—190.
3. J.A. Gouguen, L-fuzzy sets, J. Math. Anal. Appl. 18(1967), 145—174.
4. J.A. Gouguen, The Tyconoff theorem, J. Math. Anal. Appl. 43(1973), 734—742.
5. G.I. Chae and D.W. Lee, Feebly closed sets and feeble continuity in topological spaces, Indian Math. J. Pure and Appl. (submitted).
6. G.I. Chae and J.Y. Lee, A fzzy feebly open set in fuzzy topological spaces, UOU Report (submitted).
7. O. Njastad, A note on nearly open sets, Pacific J. Math., 15(1965), 961—970.
8. G.J. Nazaroff, Fuzzy topological polysystems J. Math. Anal. Appl., 41(1973), 478—485.
9. R.H. Warren, Continuity of mappings on fuzzy topological spaces, Notice A. M. S., 21(1974), A-451.
10. R.H. Warren, Neighborhoods, bases and continuity in fuzzy topological spaces, Rockey mountain J. Math., 8(1978), 459—470.
11. C.K. Wong, Fuzzy topology; Product and quotient theorems, J. Math. Anal. Appl. 45(1974), 512—521.
12. L.A. Zadeh, Fuzzy sets, Inform. and Control, 8(1965), 338—353.