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The Solution of Certain Singular Integral Equations

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〈Abstract〉

The singular integral equation of the form

$$\oint_c \frac{K(z,\zeta)}{z-\zeta}\phi(z)dz = h(\zeta)\phi(\zeta)+f(\zeta)$$

can also be solved by a more elementary method not only by a standard procedure (reducing to the Riemann-Hilbert boundary value problem) provided $K(z, \zeta)$ and $h(\zeta)$ are required to satisfy certain analyticity condition.

Here, according to the Peters' method, the solution of the singular integral equations of the form

$$\oint_c \frac{\zeta\ \phi(z)dz}{(z-\zeta)(z\zeta-1)} = i\lambda\phi(\zeta)+f(\zeta)$$

where kernels have two singular points(poles) on the unit circle is explicitly given in closed form.

어떤 특이 적분 방정식의 해법

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〈요 약〉

특이 적분방정식

$$\oint_c \frac{K(z,\zeta)}{z-\zeta}\phi(z)dz = h(\zeta)\phi(\zeta)+f(\zeta)$$

은 Kernel $K(z,\zeta)$와 $h(\zeta)$가 어떤 解析조건을 만족하면, Riemann-Hilbert 경계치 문제로 변형하지 않고서도 좀더 간단한 방법에 의하여 풀릴 수 있다.

여기서는 단위원 위에서 두개의 특이점을 갖는 함수를 Kernel로 하는 특이 적분방정식

$$\oint_c \frac{\zeta\phi(z)dz}{(z-\zeta)(z\zeta-1)} = i\lambda\phi(\zeta)+f(\zeta)$$

의 해를 Peters의 방법에 따라 구하였다.

I. Introduction

It is well known that the solution of the singular integral equation

$$\oint_c \frac{\phi(z)}{z-\zeta}dz = h(\zeta)\phi(\zeta)+f(\zeta)$$

can reduced to the Riemann boundary value problem which is explained in the texts by Muskhelishvili [1], Gakhov [2], and others. This reduction is now regared as a standard method for solving singular integral equations with Cauchy kernels.

Recently, Peters [3] has given a method to solve the integral equation of the more general form

$$\oint_c \frac{K(z,\zeta)\phi(z)}{z-\zeta}dz = h(\zeta)\phi(\zeta)+f(\zeta)$$

where $h(\zeta)$ and $K(z,\zeta)$ are required to satisfy certain analyticity condition.

He has made two claims.

(i) Under these analyticity conditions, the introduction of a Riemann-Hilbert boundary value problem is not necesary.

(ii) The method can be extended and used to solve other types of integral equation whose kernels have singular parts which are simple poles.

However, K.M.Case [4] in the criticism of Peters' paper has pointed out that the standard method does indeed yield an exact solution for the equations discussed by Peters. He also has shown that it is not the method which yields a simple result but rather the very stringent conditions of analyticity which are imposed on the given functions.

In this paper, we consider an exteneded equation of the form

$$\oint_{c\ |z|=1} \frac{\zeta\phi(z)dz}{(z-\zeta)(z\zeta-1)} = i\lambda\phi(\zeta)+f(\zeta),$$

where the kernel $K(z,\zeta)=\frac{\zeta}{z\zeta-1}$ has a singularity on C. We here solve the equation by Peters' method without reducing to the Riemann boundry value problem.

The given equation is obtained from the equation

$$\int_0^{2\pi} \frac{\phi(\theta)d\theta}{\cos\theta-\cos\omega} = \lambda_1\phi(\omega)+f_1(\omega)$$

by using the substitution $e^{i\theta}=z$ and $e^{i\omega}=\zeta$.

Since $K(z,w)=\frac{w}{zw-1}$ is analytic in each variable if $|z|<1$ and $|w|<1$, by using only the Plemelj formulas and the theory of residues, we obtained the solution in closed form.

Ⅱ. Solution by Peters' Method

We consider the equation for the unknown function φ

$$\oint_c \frac{\zeta\phi(z)}{(z-\zeta)(z\zeta-1)}dz = i\lambda\phi(\zeta)+f(\zeta) \qquad (1)$$

Here:

(i) C is the unit cirele $|z|=1$ dividing the complex plane into an interior region D^- and an exterior region D^-. The integration is taken such that D^+ always lis to the lift of C and the integral is understood in the sense of Cauchy principal value.

(ii) Each of $f(\zeta)$ and $\phi(\zeta)$ satisfies a uniform Holder condition on C

(iii) $\zeta\neq\pm1$ on C and λ is real

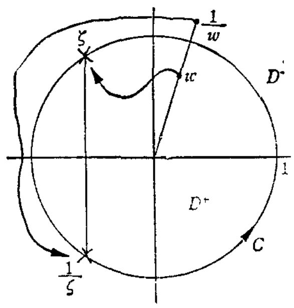

Fig. 1

We introduce a function

$$F(w)=\oint_c \frac{w\phi(z)dz}{(z-w)(zw-1)} \qquad (2)$$

analytic for w in D^+

Using the partial fraction decomposition

$$\frac{w}{(z-w)(zw-1)} = \frac{w}{w^2-1}\left\{\frac{1}{z-w} - \frac{1}{z-\frac{1}{w}}\right\}$$

(2) becomes

$$F(w)=\frac{w}{w^2-1}\left[\oint_c \frac{\phi(z)}{z-w}dz - \oint_c \frac{\phi(z)}{z-\frac{1}{w}}dz\right]$$

The limit value of $F(w)$ as w approaches a point ζ on C in a nontangential direction from D^+(and so $\frac{1}{w}$ tends to $\frac{1}{\zeta}$ from D^-) (See Fig 1.) is given by the Plemelj formulas

$$F^+(\zeta)=\frac{\zeta}{\zeta^2-1}\left[\pi i\phi(\zeta)+\oint_c\frac{\phi(z)}{z-\zeta}dz+\pi i\phi\left(\frac{1}{\zeta}\right)-\oint_c\frac{\phi(z)}{z-\frac{1}{\zeta}}dz\right]$$

$$=\frac{\pi i\zeta}{\zeta^2-1}\left[\phi(\zeta)+\varphi\left(\frac{1}{\zeta}\right)\right]+\oint_c\frac{\zeta\phi(z)dz}{(z-\zeta)(z\zeta-1)}.$$

Since equation (1) is to be satisfied, this limit value is

$$F^+(\zeta)=\frac{\pi i\zeta}{\zeta^2-1}\left[\phi(\zeta)+\phi\left(\frac{1}{\zeta}\right)\right]+i\lambda\phi(\zeta)+f(\zeta) \quad (3)$$

On the other hand, we note that eq. (1) has much symmetey in ζ with respect to the unit circle C. In fact, replacing ζ by $\frac{1}{\zeta}$ in eq. (1) we obtain

$$\oint_c\frac{\phi(z)dz}{(z-\frac{1}{\zeta})(z-\zeta)}=i\lambda\phi\left(\frac{1}{\zeta}\right)+f\left(\frac{1}{\zeta}\right)$$

Comparing with the original form of eq. (1) gives

$$i\lambda\phi(\zeta)+f(\zeta)=i\lambda\phi\left(\frac{1}{\zeta}\right)+f\left(\frac{1}{\zeta}\right)$$

or $$\phi\left(\frac{1}{\zeta}\right)=\phi(\zeta)-\frac{1}{i\lambda}\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right] \quad (4)$$

The substitution of (4) into (3) yields

$$F^+(\zeta)=\left[\frac{2\pi i\zeta}{\zeta^2-1}+i\lambda\right]\phi(\zeta)+\frac{\pi\zeta}{\lambda(\zeta^2-1)}\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right]+f(\zeta)$$

$$=\frac{i(\lambda\zeta^2+2\pi\zeta-\lambda)}{\zeta^2-1}\phi(\zeta)+\frac{\pi\zeta\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right]+\lambda(\zeta^2-1)f(\zeta)}{\lambda(\zeta^2-1)}$$

or, writing for $\varphi(\zeta)$ we obtain

$$\varphi(\zeta)=\frac{(\zeta^2-1)F^+(\zeta)}{i[\lambda\zeta^2+2\pi\zeta-\lambda]}-\frac{\pi\zeta\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right]+\lambda(\zeta^2-1)f(\zeta)}{i\lambda[\lambda\zeta^2+2\pi\zeta-\lambda]} \quad (5)$$

The subsitution of (5) into (1) gives

$$\oint_c\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{(z^2-1)F^+(z)}{i[\lambda z^2+2\pi z-\lambda]}dz-\oint_c\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{\pi z\left[f(z)-f\left(\frac{1}{z}\right)\right]+\lambda(z^2-1)f(z)}{i\lambda[\lambda z^2+2\pi z-\lambda]}dz$$

$$\equiv I_1-I_2=i\lambda\phi(\zeta)+f(\zeta) \quad (6)$$

Let C_0 be an indented counterclockwise path in D^+ shown in Fig 2. and let z_1 and z_2 be

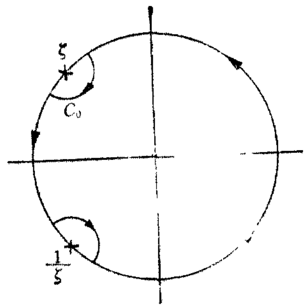

Fig. 2

the roots ($|z_1|<|z_2|$) of $\lambda z^2+2\pi z-\lambda=0$ Since $z_1z_2=-1$, z_1 is inside C_0 and z_2 is outside C_0. The deformation of C into C_0 in the first integral I_1 of (6) gives, from the residue calcalus

$$I_1=\frac{\zeta}{\zeta^2-1}\oint_c\left(\frac{1}{z-\zeta}-\frac{1}{z-\frac{1}{\zeta}}\right)\frac{(z^2-1)F^+(z)}{i[\lambda z^2+2\pi z-\lambda]}dz$$

$$=\frac{\zeta}{\zeta^2-1}\left[\pi i\frac{(\zeta^2-1)F^+(\zeta)}{i(\lambda\zeta^2+2\pi\zeta-\lambda)}-\pi i\cdot\frac{\left(\frac{1}{\zeta^2}-1\right)F^+\left(\frac{1}{\zeta}\right)}{i(\lambda\frac{1}{\zeta^2}+2\pi\frac{1}{\zeta}-\lambda)}\right]$$

$$+\oint_{c_0}\left\{\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{(z^2-1)F(z)}{i(\lambda z^2+2\pi z-\lambda)}\right]dz$$

$$=\frac{\pi_i\zeta}{\zeta^2-1}\left[\quad\right]+\oint_{c_0}\left\{\quad\right\}dz \quad (7)$$

Now, to calculate the bracket-term [ ], we use the expression (5) for $\phi(\zeta)$.

$$\varphi(\zeta)-\phi\left(\frac{1}{\zeta}\right)$$

$$=\left[\quad\right]+\frac{\pi\zeta\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right]+\lambda(\zeta^2-1)f\left(\frac{1}{\zeta}\right)}{i\lambda[\lambda\zeta^2-2\pi\zeta-\lambda]}-\frac{\pi\zeta\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right]+\lambda(\zeta^2-1)f(\zeta)}{i\lambda[\lambda\zeta^2+2\pi\zeta-\lambda]}$$

From the relation(4) and after some rearrangement we have

$$\frac{f\left(\frac{1}{\zeta}\right)-f(\zeta)}{i\lambda}=\left[\quad\right]+\frac{f\left(\frac{1}{\zeta}\right)-f(\zeta)}{i\lambda}+\pi\zeta\left[f(\zeta)+f\left(\frac{1}{\zeta}\right)\right]\frac{2\lambda(\zeta^2-1)}{i\lambda[\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2]}$$

or, $$\left[\quad\right]=-\frac{2\lambda(\zeta^2-1)\pi\zeta\left[f(\zeta)+f\left(\frac{1}{\zeta}\right)\right]}{i\lambda[\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2]}$$

Substituting this into (7) and using residue calculus for $\oint_{c_0}$, we have

$$I_1=\frac{\pi i\zeta}{\zeta^2-1}\frac{-2\lambda(\zeta^2-1)\pi\zeta\left[f(\zeta)-f\left(\frac{1}{\zeta}\right)\right]}{i\lambda[\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2]}$$
$$+\oint_{c_0}\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{(z^2-1)F(z)}{i\lambda(z-z_1)(z-z_2)}dz$$
$$=-\frac{2\pi^2\zeta^2\left[f(\zeta)+f\left(\frac{1}{\zeta}\right)\right]}{\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2}$$
$$+2\pi i\frac{\zeta}{(z_1-\zeta)(z_1\zeta-1)}\frac{(z_1^2-1)F(z_1)}{i\lambda(z_1-z_2)}$$
$$=-\frac{2\pi^2\zeta^2\left[f(\zeta)+f\left(\frac{1}{\zeta}\right)\right]}{\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2}+\frac{K\zeta}{(\zeta-z_1)(\zeta+z_2)} \quad (8)$$

where $K=\frac{-2\pi(z_1^2-1)F(z_1)}{\lambda z_1(z_1-z_2)}$ is an unknown constant. Now we need only to calculate the second integral I_2 in (6)

$$I_2=\oint_c\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{\pi z\left[f(z)-f\left(\frac{1}{z}\right)\right]+\lambda(z^2-1)f(z)}{i\lambda[\lambda z^2+2\pi z-\lambda]}dz$$
$$=\oint_c\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{[\lambda(z^2-1)+\pi z]f(z)}{i\lambda[\lambda z^2+2\pi z-\lambda]}dz$$
$$-\oint_c\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{\pi z f\left(\frac{1}{z}\right)dz}{i\lambda[\lambda z^2+2\pi z-\lambda]} \quad (9)$$

In the second integral in (9), changing the variable z by $\frac{1}{z}$ we obtain

$$\oint_c\frac{\zeta}{\left(\frac{1}{z}-\zeta\right)\left(\frac{\zeta}{z}-1\right)}\frac{\pi\frac{1}{z}f(z)d\left(\frac{1}{z}\right)}{i\lambda\left[\frac{\lambda}{z^2}+\frac{2\pi}{z}-\lambda\right]}$$
$$=\oint_c\frac{\zeta}{(z-\zeta)(z\zeta-1)}\frac{\pi z f(z)dz}{i\lambda[\lambda z^2-2\pi z-\lambda]}$$

Inserting this into (9) gives

$$I_2=\oint_c\frac{\zeta f(z)}{i\lambda(z-\zeta)(z\zeta-1)}\left[\frac{\lambda(z^2-1)+\pi z}{\lambda z^2+2\pi z-\lambda}\right.$$
$$\left.-\frac{\pi z}{\lambda z^2-2\pi z-\lambda}\right]dz$$
$$=\oint_c\frac{\zeta f(z)\lambda^2(z^2-1)^2dz}{i\lambda(z-\zeta)(z\zeta-1)[\lambda^2(z^2-1)^2-4\pi^2z^2]} \quad (10)$$

Therefore, substituting (8) and (10) into (6) and solving the result for $\varphi(\zeta)$, we obtain the solution of the given eq. (1)

$$\varphi(\zeta)=-\frac{f(\zeta)}{i\lambda}-\frac{2\pi^2\zeta^2}{i\lambda}\frac{f(\zeta)+f\left(\frac{1}{\zeta}\right)}{\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2}$$
$$+\oint_c\frac{\zeta^2(z^2-1)^2f(z)dz}{(z-\zeta)(z\zeta-1)[\lambda^2(z^2-1)^2-4\pi^2z^2]}$$

Ⅲ. Conclusion

The singular integral equation of the form

$$\oint_c\frac{\zeta\phi(z)dz}{(z-\zeta)(z\zeta-1)}=i\lambda\phi(\zeta)+f(\zeta)$$

can be solved in closed form by introducing an analytic function $F(w)=\oint_c\frac{w\phi(z)dz}{(z-w)(zw-1)}$ and using Plemelj formulas and the theory of residues, without reducing to the Riemann problem.

The solution of the given equation is

$$\varphi(\zeta)=-\frac{f(\zeta)}{i\lambda}-\frac{2\pi^2\zeta^2}{i\lambda}\frac{f(\zeta)+f\left(\frac{1}{\zeta}\right)}{\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2}$$
$$-\oint_c\frac{\zeta^2(z^2-1)^2f(z)dz}{(z-\zeta)(z\zeta-1)[\lambda^2(z^2-)^2-4\pi^2z^2]}$$

which is in accordance with Peters'.

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