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SOME THEOREMS ON STAR-OPERATION

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ABSTRACT. The aim of this paper is to study star-operation on integral domain, to characterise PVMD and Krull Domain by star-operation.

1. INTRODUCTION

Throughout this paper, we shall use R to denote integral domain with the quotient field K. An R-submodule I of K is called a *fractional ideal* of R if $dI \subseteq R$ for some nonzero element d of R. Clearly, every finitely generated R-submodule of K is a fractional ideal. Let $\mathcal{F}(R)$ be the set of all nonzero fractional ideals of R and let $\mathcal{F}_f(R)$ will denote the set of nonzero finitely generated fractional ideals of R.

A mapping $I \to I^\star$ of $\mathcal{F}(R)$ into $\mathcal{F}(R)$ is called *star-operation* on R if the following conditions hold for all $a \in K - \{0\}$ and each $I, J \in \mathcal{F}(R)$:

(1) $(a)^\star = (a)$ and $(aI)^\star = aI^\star$.

(2) $I \subseteq I^\star$ and $I^\star \subseteq J^\star$ whenever $I \subseteq J$.

(3) $(I^\star)^\star = I^\star$.

An $I \in \mathcal{F}(R)$ is called *$\star$-ideal* if $I^\star = I$ and I is called *$\star$-ideal of finite type* if there is a finitely generated ideal $J \in \mathcal{F}(R)$ such that $I^\star = J^\star$.

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We call $I \in \mathcal{F}(R)$, ideal of *strictly $\star$-finite type* if there is a finitely generated ideal $J \in \mathcal{F}(R)$ with $J \subseteq I$ and $I^\star = J^\star$.

A star operation $I \to I^\star$ is said to be of *finite type(or finite character)* if for each $I \in \mathcal{F}(R)$, $I^\star = \bigcup_\alpha \{J_\alpha^\star : J_\alpha \in \mathcal{F}_f(R)$ and $J_\alpha \subseteq I\}$. Given any star-operation $I \to I^\star$ on R, we can define a new function $I \to I^{\star_s}$ on $\mathcal{F}(R)$ by $I^{\star_s} = \bigcup_\alpha \{J_\alpha^\star : J_\alpha \in \mathcal{F}_f(R)$ and $J_\alpha \subseteq I\}$. Then $\star_s$ is a finite type star operation on R and $I^\star = I^{\star_s}$ for each finitely generated ideal $I \in \mathcal{F}(R)$.

An $I \in \mathcal{F}(R)$ is said to be *$\star$-invertible* if $(IJ)^\star = R$ for some $J \in \mathcal{F}(R)$ or equivalently $(II^{-1})^\star = R$ where $I^{-1} = [R :_K I] = \{x \in K \mid xI \subseteq R\}$.

It is easy to show by Zorn's Lemma that the set of maximal $\star_s$ ideals is not empty and that maimal $\star_s$-ideal is prime.

One of the best well-known example of star-operations is *v-operation.* For $I \in \mathcal{F}(R)$ I_v is defined by $I_v = (I^{-1})^{-1} = \bigcup_\alpha \{J_\alpha \mid \{J_\alpha\}$ is principal fractional ideals of R containging $I\}$. A v-ideal also called *divisoral(or reflexible).* But v-operation has no finite character.

Another star operation that will play an important role in this thesis is t-operation. For $I \in \mathcal{F}(R)$ I_t is defined by $I_t = \bigcup \{J_v : J \in \mathcal{F}_f(R)$ and $J \subseteq I\})$. In particular, if I is finitely generated fractional ideal of R, $I_t = I_v$. Of course, t-operation has finite character.

A simple example of star operations on R is the identity function; this star operation is called *d-operation*, $I_d = I$ for each $I \in \mathcal{F}(R)$. Clearly,d-operation has finite character.

We can note that for any star operation $\star$ on R and $A \in \mathcal{F}(R)$ $A \subseteq A^{\star_s} \subseteq A^{\star} \subseteq A_v$ [7, chapter 32, 34]. In this chapter, we shall introduce the definition of PVMD. And we shall characterize PVMD's in terms of t-invertible t-ideal. This approach is not only yields new results but also offers a unified treatment of several old ones. We give new characterization of PVMD, in terms of t-invertible t-ideal.

Also we study a Krull domain and related topics.

2. PRELMINARIES

In this section, we investigate some basic properties of star-operations for the reserch of this paper.

At first we collect some of the facts about $\star$-operations for easy references.

Proposition 2.1. *Let R be an integral domain and let $A \to A^\star$ denote a $\star$-operation. Then*

(1) $(\sum A_\alpha)^\star = (\sum A_\alpha^\star)^\star$ *for every subset* $\{A_\alpha\}$ *of* $\mathcal{F}(R)$ *for which* $\sum A_\alpha \in \mathcal{F}(R)$.

(2) $\bigcap A_\alpha^\star = (\bigcap A_\alpha^\star)^\star$ *for every subset* $\{A_\alpha\}$ *of* $\mathcal{F}(R)$ *for which* $\bigcap A_\alpha^\star \neq 0$.

(3) $(AB)^\star = (AB^\star)^\star = (A^\star B^\star)^\star$ *for every* $A, B \in \mathcal{F}(R)$.

(4) *If $\star$ is of finite type and P is a prime idael minimal over a $\star$-ideal ,then P is a $\star$-ideal*

Proof. Part (1), (2) and (3) constructed in [4, Proposition 32.2] and Part (4) is [7, Theorem 9] or [6, Proposition 1.1]

In the above Proposition (4), we obtain the important Corollary which are need in the sequel.

Corollary 2.2. *Every minimal prime ideal of principal ideal is t-ideal.*

Proof. It is clear since principal ideal is t-ideal

Proposition 2.3. *Let R be an integral domain and let $\{A_i\}$ be a set of t-ideal for which the hold ascending chain condition. Then $\bigcup\{A_i\}$ is t-ideal.*

Proof. Let $\{A_i\}$ be a t-ideal and let $A = \bigcup\{A_i\}$ which hold ascending chain condition. Clearly $A \subseteq A_t$ and A is ideal. We need only show that $A_t \subseteq A$. If $x \in A_t = (\bigcup A_i)_t$. Then, since $A_i = \{F_v \mid F \in \mathcal{F}_f(R) \text{ and } F \subseteq A_i\}$,$x \in F_v$ for some $F \in \mathcal{F}_f(R)$ such that $F \subseteq \bigcup A_i$, so $F \subseteq A_i$ for some i. But then $F_v = F_t \subseteq (A_i)_t = A_i$, so that $x \in F_v \subseteq A_i$ for some i. Hence $x \in (\cup A_i) = A$. Therefore $A_t = A$.

It is well known that a nonzero ideal I of R is invertible if and only if I is finitely generated ideal and IR_M is principal(locally principal) for each mximal ideal M of R.[10, Theorem 62]. This can be easily generalized by using t-operation compare to [9, Proposition 2.6].

Proposition 2.4. *Let R be an integral domain, then A is t-invertible if and only if AR_M is a principal ideal for each maximal t-ideal M and A is a finite type v-ideal.*

3. Results

An integral domain R is said to be a *Prüfer v-multiplication domain* (PVMD) if the set $H(R)$ of v-ideals of finite type forms a group under the v-multiplication defined by $A, B \in \mathcal{F}(R)$.

Note that in general $H(R)$ need not form a group. On the other hand, it is easy to show that $Inv_t(R)$ of t-invertible t-ideals forms a subgroup of PVMD.

We collect some of the facts about PVMD which we shall neeed in sequel for easy references.

Proposition 3.1. *Let R be an integral domain. Then the following statements are all equivalent:*

(1) *R is a PVMD.*

(2) *Every finite type v-ideal is t-invertible.*

(3) *R_M is a valuation domain for each maximal t-ideal M.*

Proof. The equivalence of (1) to (3) is due to Griffin [5, Theorem 5].

In the following Theorem, we shall give a new characterization of PVMD in terms of t-invertible t-ideal.

Theorem 3.2. *Let R be an integral domain with quotient field K. Then the following conditions are all equivalent:*

(1) *For any $a, b \in R - 0$, $aR \cap bR$ is t-invertible.*

(2) *For any $A, B \in Inv_t(R)$, $A \cap B$ is t-invertible.*

(3) *Foe any $a, b \in R - 0$, $(a) : (b)$ is t-invertible.*

(4) *The group $Inv_t(R)$ of t-invertible t-ideal of R is a lattice ordered group under the partial order $A \leq B$ if and only if $B \subseteq A$.*

(5) *Every finite type v-ideal is t-invertible.*

Proof.

(1) $\Rightarrow$ (2) Let A and B be t-invertible t -ideal. We will show that $A \cap B$ is t-invertible. By Proposition 2.4 ,we need only show that $A \cap B$ is of finite type v-ideal and $(A \cap B)_M$ is principal for any maximal t-ideal M. Then A and B are v-ideal of finite type since A and B are t-invertible t-ideal. Let $A = (a_1, a_2, \cdots, a_n)_v$ and $B = (b_1, b_2, \cdots, b_n)_v$ with $a_i \in A$ and $b_i \in B$. Since A_M and B_M are principal for each maximal t-ideal M we have $A_M = (a_0)_M$ and $B_M = (b_0)_M$ for some $a_0 \in A$ and $b_0 \in B$. Hence

$$(A \cap B)_M = A_M \cap B_M = (a_0)_M \cap (b_0)_M = ((a_0) \cap (b_0))_M.$$

By hypothesis and proposition 2.4, $(A \cap B)_M$ is principal. Finally, we will show that $A \cap B$ is a finite type v-ideal. Obviously, $A \cap B$ is a v-ideal of R. Since

$$A \cap B = (A_v \cap B_v) = (A_v \cap B_v)_v = (A \cap B)_v$$

we get $A \cap B$ is v-ideal of R. Moreover,

$$\begin{aligned}
(A \cap B)_M = A_M \cap B_M &= (a_0)_M \cap (b_0)_M \\
&= ((a_0) \cap (b_0))_M \\
&\subseteq (\textstyle\sum (a_i) \cap (b_i))_M \\
&\subseteq ((a_1, a_2, \cdots a_n) \cap (b_1, b_2, \cdots, b_n))_M \\
&\subseteq ((a_1, a_2, \cdots a_n)_v \cap (b_1, b_2, \cdots, b_n)_v)_M \\
&\subseteq (A \cap B)_M
\end{aligned}$$

Hence $A \cap B = \sum (a_i) \cap (b_i)$. By hypothesis, $(a_i) \cap (b_i)$ is t-invertible and also $(a_i) \cap (b_i)$ is strictly v-finite. So $\sum (a_i) \cap (b_i)$ is strictly v-finite.

Hence $A \cap B$ is finite type v-ideal.

Therefore $A \cap B$ is t-invertible.

(2) $\Rightarrow$ (1). Since principal ideal is t-invertible t-ideal, the statement is clear.

(1) $\Leftrightarrow$ (3). The equivalence follows from the fact that $(a) \cap (b) = ((a) : (b))(b)$ for $a, b \in R$

(2) $\Rightarrow$ (4). We note that if $A, B \in Int_t(R)$, then $sup(A, B)$ exists if and only if $A \cap B \in Inv_t(R)$, and so $sup(A, B) = A \cap B$. Therefore $Inv_t(R)$ is a lattice ordered group [4, Theorem 15.4].

(2) $\Rightarrow$ (5). Let A be a finite type v-ideal. Then $A = B_v$ for some finitely generated ideal B of R. Let $B = (b_1, b_2, \cdots, b_n)$ where $b_i \in K$. Then

$$A = B_v = ((b_1, b_2, \cdots, b_n)^{-1})^{-1}.$$

But then by hypothesis

$$(b_1, b_2, \cdots, b_n)^{-1} = ((1/b_1)R \cap (1/b_2)R \cdots \cap (1/b_n)R)$$

is t-invertible. Therefore $A = B_v$ is t-invertible.

(5) $\Rightarrow$ (4). If $A, B \in Inv_t(R)$, then $A = (a_1, a_2, \cdots, a_n)_v$ and $B = (b_1, b_2, \cdots, b_n)_v$. Then

$$(A + B)_v = ((a_1, a_2, \cdots, a_n)_v + (b_1, b_2, \cdots, b_n)_v)_v$$

$$= ((a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n))_v$$

The third equality follows from Proposition 2.1 (1). So $(A + B)_v$ is finite type v-ideal. Therefore $(A + B)_v$ is t-invertible by hypothesis. Thus $Inf(A, B) = (A + B)_v$. Therefore $Inv_t(R)$ is a lattice orderd group [1, Theorem 15.4].

With the help of Proposition 3.1 and Theorem 3.2, we obtained a new characterization of PVMD.

Corollary 3.3. *Let R be an integral domain. Then the following statements are all equivalent:*

(1) *R is a PVMD.*

(2) *For any $a, b \in R - 0$, $aR \cap bR$ is t-invertible.*

(3) *For any $A, B \in Inv_t(R)$, $A \cap B$ is t-invertible.*

(4) *Foe any $a, b \in R - 0$, $(a) : (b)$ is t-invertible.*

(5) *The group $Inv_t(R)$ of t-invertible t-ideal of R is a lattice ordered group under the partial order $A \leq B$ if and only if $B \subseteq A$.*

(6) *Every finite type v-ideal is t-invertible.*

The following Proposition has been obtained by S. Gabelli and Zafrullah.

Proposition 3.4 [13]. *If P is a t-invertible prime t-ideal of R, then*

(1) *P is a minimal prime of a principal ideal.*

(2) *P has the form $[(a) : (b)]$ for some $a, b \in R$.*

(3) *P is a maximal t-ideal.*

We collect some equivalent conditions for Krull domains.

Proposition 3.5. *The following condition are all equivalent:*

(1) *R is a Krull domain.*

(2) *R is a completely integrally closed Mori domain.*

(3) *Every $A \in \mathcal{F}(R)$ is t-invertible.*

(4) *Every prime ideal of R is t-invertible.*

(5) *Every associated prime ideal of R is t-invertible.*

Proof. The reader can refer to [3, Theorem 3.6] for the equivalence of (1) and (2) , to [11, Theorem 2.5] for the equivalence of (1) to (5).

In [1], D.D.Anderson prove that if R is locally UFD then R is π-domain if and only if every nonzero minimal prime ideal is a finitely generated ideal.

Recall that R is a π-domain if every principal ideal is product of prime ideals or equivalently every minimal prime ideal over principal ideal of R is invertible. In this thesis, we show the following generalizationof the above mentioned result of D.D. Anderson.

Let R be a t-locally UFD. Then every principal ideal has finitely many maximal t-ideal if and only if R is Krull domain.

We recalled that an integral domain R is a *t-locally UFD* if for each maximal t-ideal M, R_M is an UFD.[9]

Theorem 3.6. *Let R be an integral domain. The following statements are all equivalent:*

(1) *R is a Krull domain.*

(2) *R is a t-locally UFD and every minimal prime ideal is finite type v-ideal.*

(3) *R is a t-locally UFD and every minimal prime ideal over principal ideal is finite type v-ideal.*

(4) *R is a t-locally UFD and every principal ideal has finitey many maximal t-ideal.*

Proof.

(1) $\Rightarrow$ (2). Assume that R is Krull domain. According to the Proposition 3.5 and the proposition 2.4 , every minimal prime ideal is finite type v-ideal. We next show that R_M is an UFD for each maximal t-ideal M. Let P be a prime ideal of R. By Proposition 3.5 and Proposition 2.4, PR_M is principal. Therefore R_M is an UFD.

(2) $\Rightarrow$ (3). This is Clear.

(3) $\Rightarrow$ (4). We claim that every principal ideal has finitely many maximal t-ideal. We first show that every principal ideal has finitely many mimal prime ideal. Let $\mathcal{C} = \{P_1P_2\cdots P_n \mid \text{each } P_i \text{ is a minimal prime } t\text{-ideal over } (a)\}$. It suffices to show that $K \subseteq (a)$ for some $K = P_1P_2\cdots P_n \in \mathcal{C}$. For if Q is a minimal prime ideal over (a), then $P_1P_2\cdots P_n \subseteq Q$. Since Q is prime ideal and $P_1P_2\cdots P_n \subseteq Q$ it follows that $P_i \subseteq Q$ for some P_i and therefore $P_i = Q$ by minimality. Thus $\{P_1, P_2, \cdots, P_n\}$ is the set of minimal prime ideal over (a). Therefore every principal ideal has finitely many minimal prime ideal. Assume that $K \not\subseteq (a)$ for some $K \in \mathcal{C}$. We consider the set $\mathcal{T} = \{I_\alpha \mid I_\alpha \text{ is } t\text{-ideal of } R \text{ with } (a) \subseteq I_\alpha \text{ and } K \not\subseteq I_\alpha \text{ for each } K \in \mathcal{C}\}$, then $\mathcal{T} = \emptyset$ since $(a) \in \mathcal{T}$. The family $\mathcal{T}$ partially orderd by inclusion is inductive.

For if L is chain in $\mathcal{C}$ and $A = \cup\{A_i \mid A_i \in L\}$, then A is t-ideal by Lemma 4.1. Hence L has upper bound in $\mathcal{C}$. Thus by Zorn,s Lemma, $\mathcal{T}$ has maximal element Q. Clearly Q is a prime t ideal. But then $(a) \subseteq Q$ can be shrunk to prime t ideal Q_0 minimal over (a). Hence $Q_0 \in \mathcal{C}$. This is a contradiction. Finally we show that every minimal prime ideal is maximal t-ideal. Let P be a minimal prime ideal over (a). By Corollary 2.2, P is t-ideal. We note that PR_M is principal for each maximal t-ideal. We consider two cases. Case 1) If $P \not\subseteq M$ for some maximal t-ideal,then ,since $PR_M = R_M$, PR_M is principal. Case 2) If $P \subseteq M$ for all maximal t-ideal,then, since R_M is UFD and P_M is a minimal prime ideal over $(a)_M$, P_M is principal. Therefore P_M is a principal for all maximal t-ideal. By asssumption, P is finite type v-ideal. By Proposition 2.4, P is t-invertible t-ideal. Therefore P is maximal t-ideal by proposition 3.4.

(4) $\Rightarrow$ (1). Let P be a minimal prime ideal over (a) of R and M be any maximal t-ideal. We show that P is t-invertible. By similar argument of proof (3) $\Rightarrow$ (4), P_M is principal for all maximal t-ideal. Then by [1,Lemma 26.2 and Lemma 3.4], P is t-invertible. Therefore R is a Krull domain.

REFERENCES

[1] D. D. Anderson , *Graded π-domain* , Canad. J. Math 449–457. 1978,

[2] V.Barucci and S. Gabelli, *How far is a Mori domain from being a Krull domain ?*, Journal of Pure Applied Algebra, Vol 45 ,101–112 ,1987

[3] R. Fossum, *The divisor class group of Krull domain*, Springer-Verlag, New York, 1972

[4] R. Gilmer *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972

[5] M. Griffin, *Some results on v-multiplication rings*, Canad. J. Math., 19, 710–712, 1967

[6] Jhon R. Hedstrom and Evan G. Houston,*Some remarks on star-operations*, Journal of Pure and Applied Algebra, Vol 18, 37–44, 1980

[7] P. Jafford, *Les Systems d'Ideaux*, Dunod, Paris, 1960

[8] B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J of Algebra, 123(1), 151–170, 1989

[9] B.G. Kang , *On the Converse of well-known Fact about Krull Domain*, J of Algebra 124, 284-299 , 1989

[10] I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, 1970

[11] J.L.Mott and M. Zafrullah, *On Krull domains*,Arch. Math. Vol 56, 559–568, 1991

[12] C.H.Park, *A Study on star operations and related Topics*, Doctoral Disseration, University of Ulsan,1997

[13] M. Zafrullah, *On finite conductor domains*, Manuscripta Math., Vol 24, 191–204, 1978

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