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On the Results of Invariant Sets Under Translation for Probability Measure

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〈**Abstract**〉

Let $\hat{\mu}(t)$ denote the Fourier Transform of probability measure. If $s(\mu)=\{t \mid \hat{\mu}(t)=0\}$ consists of finitely many elements then μ is weakly complete. We shall show that if $S(\mu)$ is compact then μ is weakly complete.

확률측도의 변환에서 불변인 집합의 결과에 대하여

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〈요 약〉

Probability measure μ의 characteristic 함수가 0이 되는 집합이 유한이면 weakly complete하다는 것이 이미 입증 되었지만 그 결과를 compact 상에서도 성립함을 증명한다.

I. Introduction.

The main object of this paper is to present some results which we have recently obtained concerning measure invariant sets for translation parameter family of probability measure.

The study of measure invariant sets was initiated by Basu and Ghosh. They proved a number of interesting results concerning these sets and posed some unsolved problems. Our main object is to make a careful study of some of the conjectures contained in their paper.

Basu and Ghosh show that if $S(\mu)=\{t \mid \hat{\mu}(t)=0\}$ where the characteristic function of μ vanishes consists of finitely many elements, then μ is weakly complete. We shall strengthen some of their results and show that if $S(\mu)$ is compact then μ is weakly complete.

II. Preliminaries

Let μ be a given probability measure on (R, B), where R is the real line and B is the family of Borel sets on R. A measurable set A is called measure-invariant (μ-invariant) if $\mu(A+\theta)=\mu(A)$ for any θ, $-\infty<\theta<\infty$ where $A\in B$ and $A+\theta=\{X+\theta \mid x\in A\}$. We denote by $A(\mu)$ the family of all μ-invariant sets. For exmple the null set ϕ and the whole space are trivially μ-invariant set. A set A is called nontrivial if $0<\mu(A)<1$. The probability measure μ is called weakly incomplete(weakly complete) if it does (does not) prosses a nontrivial μ-invariant set. Consider $L^1(R,B,\lambda)$, where λ denote the lebesque measure. For every $f\in L^1(R, B, \lambda)$, or simply L^1, $I(f)$ denote the ideal generated by

f and $s(f)=\{t|\hat{f}(t)=0\}$ where $\hat{f}(t)=\int \exp(itx)\, df(x)$ denote the Fourier transform of f.

The following theorems are then well known. (see, Rudin)

Theorem (1) If $f,\ g\in L^1$ and $s(f)\subset s(g)$ and if the intersection of the boundaries of $s(f)$ and $s(g)$ is contable, then $g\in I(f)$.

Theorem (2) Let K be a compact set. Then there extists a non-trivial bounded function $h\in L^1$ such that $\hat{h}(x)=1$ if $x\in K$.

Theorem (3) Let $f\in L^1$, let $u\in L^\infty$ and suppose that $u*f=0$ Then $u*g=0$ for every $g\in I(f)$.

Theorem (4) (Basu-Ghosh)

Let $f\in L^1$, $s(f)=\{\pm c, \pm 2c, \cdots\}$. Let $g\in L^\infty$ and suppose that $g*f=\alpha$, where α is a constant. Then $g(X+2\pi/c)=g(x)$.

Theorem (5) (Basu-Ghosh)

Let μ and ν be two probability measures. Suppose that $A\in A(\mu)$. Then $A\in A(\mu*\nu)$ and $\mu*\nu(A+\theta)=\mu(A)$ for every $\theta\in X$.

Theorem(6) Let $f\in L^1$ and $s(f)$ be compact. Let $g\in L^\infty$ and suppose g assumes finitely many values. If $f*g=0$, then $g=0$.

proof) Let K be a compact set such that $\text{int}(k)\supset s(f)$. From Theorem 2, there exists non-trivial bounded function $h\in L^1$ such that $\hat{h}(x)=1$ for every $x\in K$. Let $k\in L^1$ and consider $l=k*(1-h)$ so that $S(l)\supset k\supset \text{int}(k)\supset s(f)$. By theorem 1, it follows that $l\in I(f)$. Consequently $g*l=0$ so that $g*l=g*k*(1-h)=k*(g-g*h)=0$ for any $k\in L^1$ Hence $g=g*h$, it follows that g and $g*h$ have the same essential range. Since R is connected and $g*h$ continous, it follows that the essential range of $g*h$ and g is connected set. The hypothesis that assumes finitely many values that g assumes finitely many values and $g*f=0$ imply that $g=0$.

III. Main Results

DEFINITION (7) Let (X, B) be measurable space, and μ and ν are two measures defined on (X,B). A measure ν is said to be absolutely continuous with respect to measure μ if $\nu(A)=0$ for each set A for which $\mu(A)=0$. We use the symbolism $\nu\ll\mu$ for ν absolutely continous with respect to μ.

Theorem (8) If $\nu\ll\mu$ and f is a non-negative measurable function then $\int f d\nu=\int f\left[\frac{d\nu}{d\mu}\right]d\mu$. The proof of above theorem is given by H.L. Royden. (7)

COROLLARY (9) Let μ_1 and μ_2 be two absolutely continuous probability measures with respect to λ with $S(\mu_1)=S(\mu_2)$. Then $S(f_1)=S(f_2)$ where $f_i=d\mu_i/d\lambda$.

$$\text{(proof)} \int \exp(itx) f_1 d\lambda=\int \exp(itx)\frac{d\mu_1}{d\lambda}d\lambda$$
$$=\int \exp(itx)\frac{d\mu_2}{d\lambda}d\lambda=\int \exp(itx) f_2 d\lambda$$

Theorem (10) Let μ_1, μ_2 be two absolutely continuous probability measures with $S(\mu_1)=S(\mu_2)$ if the boundary of $S(\mu_1)$ is countable, then $A(\mu_1)=A(\mu_2)$.

proof) Let $f_i=d\mu_i/d\lambda$ and $A\in A(\mu_i)$, Then $A\in A(\mu_1)$ iff $[I_{-A}-C]*f_1=0$ where $c=\mu_1(A)$. Since $S(f_1)=Sf_2)\,[\equiv S(\mu_1)=S(\mu_2)]$ and the boundary of $s(f_1)$ is countable, it follows from theorem 1 that $f_2\in I(f_1)$so that $[I_{-A}-C]*f_2=0$. Consequently $A\in A(\mu_2)$ and $\mu_2(A)=\mu_1(A)=C$.

Basu and Ghosh show that if μ is a probability measure, $S(\mu)$ consists of finitely many elements, then μ is weakly complete. Our next result is a strengthening of this theorem and we show that if $S(\mu)$ is compact, then μ is weakly complete.

Theorem (11) Let μ be a probability measure and suppose that $S(\mu)$ is compact. Then μ is weakly complete.

proof) First let $\mu\ll\nu$ and $f=d\mu/d\lambda$. Let $A\in A(\mu)$ with $\mu(A)=C$. Then $(I_{-A}-C)*f=0$. Since $S(f)$ is compact and by theoren 6, it follows tha $I_{-A}=C$. Hence $C=0$ or 1. In the general case let ν be absolutely continuous probability measure with $S(\nu)=\phi$. Then $A\in A(\mu*\nu)$ with $\mu(A)=\mu*\nu(A)$. But $\mu*\nu$ is absolutely continuous with $S(\mu*\nu)\,[\equiv S(\mu)]$ compact. Consequently $\mu(A)=\mu*\nu(A)=0$ or 1. This μ is weakly complete.

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