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Some notes about Hausdorff μ-space

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⟨Abstract⟩

In this paper we define a Hausdorff μ-space and study its properties. In section 2, we consider some properties between the given topology and the μ-extention topology. In section 3, we investigate the necessary condition and the sufficient condition for the Hausdorff μ-space. In section 4, we prove that a locally μ-space is a μ-space and any closed subspace of μ-spaces is a μ-space.

Hausdorff μ-space에 관한 연구

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⟨요 약⟩

본 논문에서는 Hausdorff μ-space를 정의하여 이들이 갖는 몇가지 성질들을 알아 보았다. 제 2절에서는 일반 Hausdorff인 위상공간에서 이들 보다 finer한 $\mathscr{T}_\mu$-공간을 정의하여 본래 주어진 위상공간과의 관계를 비교하였고, 제 3절에서는 Hausdorff μ-space의 필요조건과 충분조건을 찾아 보았다. 제 4절에서 국소 μ-space를 정의하여 국소 μ-space이면 μ-space임을 증명하였고 임의의 μ-space의 폐부분 공간은 μ-space임을 증명하였다.

Ⅰ. Introduction

In this paper we are concerned with a Hausdorff μ-space. By a μ-space is meant a topological space which is coherent with its metacompact closed subsets, i.e. X is a μ-space if open subsets of X are exactly those sets which intersect every metacompact closed subset M of X in an open subset of M. In section 2, we consider some properties between the given topology and the μ-extention topology. In section 3, we investigate two topological properties which are closely related to μ-space. One of them ensures a space to be a μ-space, while the other proved to be a necessary condition to become a μ-space. In section 4, we proved that locally μ-space is μ-space and every closed subset of a μ-space is a μ-space. Unless otherwise mentioned, the word "space" is used to meant a Hausdorff space.

Ⅱ. The μ-extention of a topology.

Let $(X, \mathscr{T})$ be a Hausdorff space. The μ-extention of $\mathscr{T}$ is defined to be the family $\mathscr{T}_\mu$ of all subsets U of X such that $U \cap M$ is open in M for every metacompact closed subset M

Proposition 2.1. If M is a $\mathscr{T}$-metacompact closed subset of X, then the relativization of $\mathscr{T}$ to M is identical with that of $\mathscr{T}_\mu$. Consequently a set is $\mathscr{T}$-metacompact closed subset iff it is $\mathscr{T}_\mu$-metacompact closed subset.

Proof. Suppose M is $\mathscr{T}$-metacompact closed subset. Since M is $\mathcal{T}_\mu$-closed as $\mathcal{T}_\mu$ is finer than $\mathcal{T}$, a subset A of M is $\mathcal{T}_\mu$-closed in the whole space whenever it is relatively $\mathcal{T}_\mu$-closed in M. Hence a subset A of M is closed in M with respect to $\mathcal{T}_\mu$, then its intersection with any metacompact closed subset must be $\mathcal{T}$-closed. $A=A\cap M$ is $\mathcal{T}$-closed, and $\mathcal{T}$ is finer than $\mathcal{T}_\mu$ on M.

Proposition 2.2. The space $(X, \mathcal{T}_u)$ is a μ-space.

Proposition 2.3. A functionon X is $\mathcal{T}_\mu$-continuous iff it is $\mathcal{T}$-continuous on every metacom-pact closed subset of X.

Proof. If f is continuous on every metacompact closed subset, then $(f|M)^{-1}(C)=f^{-1}(C)\cap M$ is $\mathcal{T}$-closed for C closed in Y whenever M is a metacompact closed subset. Hence $f^{-1}(C)$ is closed with respect to $\mathcal{T}_\mu$, i.e. f is $\mathcal{T}_\mu$-continuous. Since $\mathcal{T} = \mathcal{T}_\mu$ on every metacompact closed subset, the converse is true.

Proposition 2.4. The topology $\mathcal{T}_\mu$ is the largest topology which agrees with $\mathcal{T}$ on metacompact closed subsets.

Proof. By proposition 2.3, the identity function of X is continuous of $(X, \mathcal{T}_\mu)$ to $(X, \overline{\mathcal{T}})$ if $\overline{\mathcal{T}}$ agrees with $\mathcal{T}$ on metacompact closed subsets.

III. Necessary condition and sufficient condition for μ-space.

Proposition 3.1. A space is a μ-space if for any subset A of X having $p\varepsilon X$ as a limit point there is a metacompact closed subset M of X such that p is a limit point of $A\cap M$.

Proof. Let A be a nonclosed subset of X. We must show that for some metacompact closed subset M the intersection $A\cap M$ is not closed. Suppose p is a limit point of A which does not belong to A. There is a metacompact closed subset M of X such that p is a limit point of $A\cap M$. But the point p cannot be contained in $A\cap M$ as p is not contained in A. Hence $A\cap M$ is not closed.

Proposition 3.2. If a space X is a μ-space, then for any closed subset C of X having p as a limit point there is a metacompact closed subset M of X with p as a limit point of $C\cap M$.

Proof. Suppose that the proposition is false, and let X be a μ-space which does not satisfy the condition that there is a metacompact closed subset M of X with p as a limit point of $C\cap M$ whenever C is a closed subset of X having $p\varepsilon X$ as a limit point. Then there is a point p and a closed subset C of X having p as a limit point such that p is not a limit point of $C\cap M$ whenever M is a metacompact closed subset of X. This implies that $(C-\{p\})\cap M=(C\cap M)-\{p\}$ is closed in X. Since X is a μ-space, $C-\{p\}$ is also a closed subset of X and p cannot be a limit point of C. This contradiction proves the proposition.

IV. Locally μ-space and free union of the spaces.

Let X be a Hausdorff space and p be a point of X. By a μ-neighborhood of p, we mean a closed neighborhood of p which is a μ-space as a subspace of X. We call a space is a locally μ-space iff every point p of X has a μ-neighborhood.

Proposition 4.1. A space is a μ-space if and only if it is a locally μ-space.

Proof. For sufficiency, let A be a non-closed subset of a μ-space X. Then there is a point p of A which is not in A. Let N be a μ-neighborhood of p. The intersection $A\cap N$ has p as a limit point with $p\not\in A\cap N$. Thus $A\cap N$ is not closed in the μ-space and there is a metacompact closed subset M of N such that $(A\cap N)\cap M$ is not closed in N. This means that $A\cap M$ is not closed since $A\cap N\cap M=A\cap M$. Necessity is obvious.

Proposition 4.2. The free union of the μ-spaces is also a μ-space.

Proof. The proof is obvious from proposition 4.1.

Proposition 4.3. Every closed subset of μ-space is a μ-space.

Proof. Let C be a closed subspace of a μ-space, and let S be a subset of C such that $S \cap M$ is closed in M whenever M is a metacompact closed subset of C. Then, for any metacompact closed subset N of X, $(S \cap C) \cap N = S \cap (N \cap C)$ is closed in $N \cap C$ as $N \cap C$ must be a metacompact closed subset of C. Since $N \cap C$ is closed in N, $(S \cap C) \cap N$ is closed in N. This means that $S \cap C$ is closed in X. Therefore S is closed in C.

Reference

1. James Dugundji. Topology, Allyn and Bacon, INC., Boston (1966) p. 160~180.
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