UIT Report Vol. 11, No. 2, pp. 179~185, 1980　　연구논문집 제11권 제2호 pp. 179~185, 1980

A Branch and Bound Algorithm for the Multi-period Capacitated Facilities Location Problem*

Choi, Won-Joon
Dept. of Industrial Engineering
(Received June 10, 1980)

〈**Abstract**〉

This paper is concerned with the problem of locating warehouses in a two-stage distribution system, where commodities are delivered from capacitated plants to customers via capacitated warehouses over a relatively long planning horizon. We want to decide which of a set of warehouses to open, when to open them, and how to allocate the production of each plant to satisfy the known demands to minimize the total discounted costs.

This problem is formulated as a mixed integer linear programming model. To obtain the optimal solution of this model, a new branch and bound algorithm is presented. To reduce the size of branch and bound tree and the total computation time, four kinds of node simplification steps are devised through exploiting the economic charateristics of the model.

유통설비의 최적입지 선정에 관한 연구*

최 원 준
산 업 공 학 과
(1980.6.10 접수)

〈**요 약**〉

本연구는 다계획기간동안 2단계 유통구조內에서의 창고의 입지선정을 다루고 있다. 2단계 유통구조란 物的 흐름이 공장에서 창고를 거쳐 소비지에 이르는 유통구조를 말한다. 여기서는 모든 공장과 창고의 생산 혹은 취급능력이 有限인 경우를 대상으로 한다.

계획기간동안의 모든 수요를 충족시키면서 관련 총 비용을 최소로 하는 창고의 입지를 선정하기 위해, 이를 혼합 정수계획모형으로 정식화하고 이 모형의 최적해를 구하기 위한 새로운 分段探索法(branch and bound algorithm)을 제시하였다. 이때 分段探索나무(*b* & *b* tree)의 크기를 줄이기 위해 4가지 유형의 마디의 간략화단계(node simplification steps)를 보였다.

I. Introduction

This paper is concerned with a special case of the general facility location problem - the problem of locating warehouses in a two-stage distribution system where commodities are delivered from plants to customers via warehouses. In the management of distribution systems is commonly faced the problem of determining a set of geographical warehouse locations such that demands are satisfied with a minimum total discounted costs over a relatively long planning period.

Considering the real-world importance of a commonly occuring problem of locating intermediate storage centers(warehouses) between plants and customers, only a few studies have been devoted to the two-stage location problems.

* This research was supported by the U.I.T. Hyundai Research Scholarships in 1979.

Surveys on the single-period two-stage warehouse location problem can be found in Choi [2] and Geoffrion [5]. On the multi-period problem, El-Shaieb [3], Eschenbach and Carlson [4] and Roodman and Schwarz [6], [7] dealt with the cases in a single-stage distribution system. Sweeney and Tatham [8] approached the multi-period capacitated warehouse location problem by applying the dynamic programming algorithm to the model formulated in mixed integer linear problem. Choi [2] dealt with the single period capacitated warehouse location problem in a two-stage distribution system by applying the branch and bound routine.

In this paper, extending the works of Choi [2] to a T time period planning horizon model, a new branch and bound algorithm for this model will be presented.

Ⅱ. The Model

The assumptions of the model are the following.

1. The total discounted costs consist of : variable operating costs at each plant and warehouse; variable transportation costs between plants, warehouses and customers; and fixed costs of opening and operating warehouses.
2. Demands are known over time and must be met on time with no inventories allowed.
3. The locations of plants and customers are given fixed.
4. There is a capacity constraint on each plant and warehouse.
5. Warehouses can't be closed once opened. This assumption decreases the number of possible integer combinations for given problem dimensions.
6. Time periods may be of any length.

The following definitions will be used in the mathematical model.

l: number of plants

m: number of potential warehouses

n: number of customers

T: number of time periods

S_{it}: production capacity at plant i during period t

W_{jt}: capacity of warehouse j during period t

D_{kt}: demand of customer k during period t

C^1_{ijt}: discounted production and transportation cost per unit commodity shipped from plant i to warehouse j during period t

C^2_{jkt}: discounted handling and transportation cost per unit commodity shipped from warehouse j to customer k during period t

F_{jt}: discounted fixed cost incurred if warehouse j is opened during period t, rather than during period $t+1$

x^1_{ijt} : amount shipped from plant i to warehouse j during period t

x^2_{jkt}: amount shipped from warehouse j to customer k during period t

y_{jt}: 0−1 variable that will be be 1 if warehouse j is open during period t and 0 otherwise.

<u>The Mathematical Model</u>

$$\text{minimize } Z=\sum_{ijt} C^1_{ijt}\, x^1_{ijt} + \sum_{jkt} C^2_{jkt} x^2_{jkt} + \sum_{jt} F_{jt} y_{jt} \qquad (1)$$

$$\text{subject to } \sum_j x^1_{ijt} \leq S_{it}, \quad i=1,\cdots,l \quad t=1,\cdots,T \qquad (2)$$

$$\sum_i x^1_{ijt} \leq W_{jt}\, y_{jt}, \quad j=1,\cdots,m \quad t=1,\cdots,T \qquad (3)$$

$$\sum_i x^1_{ijt} - \sum_k x^2_{jkt} = 0, \quad j=1,\cdots,m \quad t=1,\cdots,T \qquad (4)$$

$$\sum_j x^2_{jkt} \geq D_{kt}, \quad k=1,\cdots,n \quad t=1,\cdots,T \qquad (5)$$

$$x^1_{ijt},\, x^2_{jkt} \geq 0, \text{ for all } i,j,k \text{ and } t \qquad (6)$$

$$y_{jt} \in \{0,1\}, \quad j=1,\cdots,m \quad t=1,\cdots,T \qquad (7)$$

$$y_{jt} \leq y_{jt+1}, \quad j=1,\cdots,m \quad t=1,\cdots,T \qquad (8)$$

We will call this model P throughout this paper. Constraint(2) requires that the capacity of each plant be not exceeded; constraints (3) and (4) require that the capacity of each warehouse be not exceeded if the warehouse is open in that period and each of the warehouses conserve the commodity flow by having input equal to output; and constraint (5) requires that each of the demands must be met on time. Constraint (8) requires that warehouse can't be closed after being opened.

In what follows, we propose a branch and bound algorithm for solving the problem P. The algorithm exploits special properties of the model:

a) With the deletion of constraint(8), problem (1)—(7) decomposes into T single-period warehouse location problems; and

b) After decomposition, let $V_t[A_t]$ be the optimal value of the transshipment problem with the set of warehouses given by a set A_t during period t. Say, $V_t[A_t]$ is

$$\text{minimum } \sum_{ij} C^1_{ijt}\, x^1_{ijt} + \sum_{jk} C^2_{jkt}\, x^2_{jkt}$$

$$\text{s.t. } \sum_j x^1_{ijt} \leqslant S_{it},\ i=1,\cdots,l$$

$$\sum_i x^1_{ijt} \leqslant W_{jt},\ j \in A_t$$

$$\sum_i x^1_{ijt} - \sum_k x^2_{jkt} = 0,\ j \in A_t$$

$$\sum_{j\in A_t} x^2_{jkt} \geqslant D_{kt},\ k=1,\cdots,n$$

$$x^1_{ijt},\ x^2_{jkt} \geqslant 0, \text{ for all } i,j \text{ and } k.$$

Then the following key property of the model P holds. For proof, see Choi [2].

<u>Property 1</u>

Suppose $B_t \subseteq A_t \subseteq \{1,\cdots,m\}$. Then for any $j \in \{1\cdots,m\} - A_t$,

$$V_t[A_t] - V_t[A_t \cup \{j\}] \leqslant V_t[B_t] - V_t[B_t \cup \{j\}].$$

The physical meaning of property 1 is that the savings in the variable cost due to having a particular warehouse open becomes the smaller as the set of the existing warehouses gets the larger.

From here on, we shall denote $V_t[A_t]$ by $V[A_t]$ for the sake of exposition. And for future use, define

$$Z[A_t] = V[A_t] + \sum_{j\in A_t} F_{jt}.$$

Ⅲ. The Branch and Bound Algorithm

The presented procedure for solving the problem P is a kind of branch and bound (b & b) procedures which generally require generating and analyzing a sequence of candidate problems in search of successively better feasible solutions. A candidate problem, denoted by CP, is the problem P or the restricted version of P in which the values of some 0—1 variables y_{jt}'s are specified. For a CP, let K_{0t}, K_{1t} and K_{2t} for $t=1,\cdots,T$ denote the sets containing the j-indices of y_{jt}'s which are constrained to be 0(closed), 1(open), and unconstrained (free) during period t respectively. Any assignment formed from candidate problem CP by specifying feasible values of all y_{jt} for all $j \in K_{2t}$, $t=1,\cdots,T$, will be termed a completion of CP.

The process of b & b procedure can be shown schematically as a tree, called a b & b tree. In a b & b tree, each node corresponds to a CP. At the initial node, obviously $K_{0t}=K_{1t}=\phi$ and $K_{2t}=\{1,\cdots,m\}$ for $t=1,\cdots,T$. For each candidate problem CP, we also define E_j and L_j as follow:

E_j: first period in which warehouse j may be open in any completion of this node

L_j: last period in which warehouse j may be closed in any completion of this node.

From these definitions, we can see that if $t<E_j$ then y_{jt} must be in K_{0t}, and if $t>L_j$ then y_{jt} must be in K_{1t}.

A given node CP is fathomed by one of several results: it is solved; it is shown to have no feasible solution; or it is shown to have no feasible solution with a smaller objective value than the incumbent value $\bar{Z}$(i.e., the objective value of the best solution found so far).

It a CP can't be fathomed, a free variable y_{jt}, $j \in K_{2t}$, $t=1,\cdots,T$ is chosen to be constrained, and the CP is seperated into two new successor nodes corresponding to the two restrictions $y_{jt}=0$ and $y_{jt}=1$. This is termed as branching. But if at any node CP it can be shown that a free variable y_{jt}, $y \in K_{2t}, t=1,\cdots,T$ is sure to be 1 (or 0) in the best completion from this CP, the branch $y_{jt}=0$(or 1) can be pruned directly at this ode before taking the branching step. This is eferred to as node simplification, or as fixing warehouse open (or closed) to distinguish from onstraining a warehouse due to branching. The node simplification steps, if used efficiently, may reduce the size of b & b tree and the total computation time.

1. The Node Simplification Steps Ⅰ & Ⅱ

If $\bigcup_{t=1}^{T} K_{2t}$ is empty at a node, the node is called a terminal node. Otherwise it is called a non-terminal node. Four node simplification steps (NSS) Ⅰ, Ⅱ, Ⅲ and Ⅳ applicable to every non-terminal node of the b & b tree will be presented. From here on, the abbreviation of NSS will be used to denote the node simplification step. NSS Ⅰ and Ⅳ are sufficient conditions for a free warehouse to be fixed open, whereas NSS Ⅱ and Ⅲ to be fixed closed. NSS Ⅲ and Ⅳ will be presented later.

At a node of b & b tree, for $j \in K_{2t}$, $t=1, \cdots, T$, define

$$\Delta_{jt} = V[K_{1t} \cup K_{2t} - \{j\}] - V[K_{1t} \cup K_{2t}]$$
$$\Omega_{jt} = V[K_{1t}] - V[K_{1t} \cup \{j\}]$$
$$A_{jt} = \sum_{p=t}^{L_j} (\Delta_{jp} - F_{jp})$$
$$B_{jt} = \sum_{p=E_j}^{t} (\Omega_{jp} - F_{jp}).$$

From the property 1, the following is easily derived.

Property 2

a) Δ_{jt}(or Ω_{jt}) repesents a lower bound(or upper bound) on the savings in variable costs due to opening the warehouse j during period t without considering the restriction(8) of the model P.

b) Δ_{jt}(or Ω_{jt}) is nondecreasing (or nonincreasing) from a node to its successor nodes.

Simplification Step Ⅰ

At a node of b & b tree, determine

$$A_{jf} = \max_{E_j \le t \le L_j} A_{jt} \quad \text{for} \quad j \in \bigcup_{t=1}^{T} K_{2t}.$$

If $A_{jf} > 0$, then there exists a best completion of this node such that $y_{jt}=1$ for $f \le t \le L_j$.

Proof: Suppose the above is not true. Then this would mean that there exists a $\hat{t}$, $f < \hat{t} \le L_j$ such that $y_{jt}=0$ for $E_j \le t < \hat{t}$ and $y_{jt}=1$ for $t \ge \hat{t}$. By definition, $A_{jf} \ge A_{j\hat{t}}$. Thus

$$\sum_{t=f}^{L_j} (\Delta_{jt} - F_{jt}) \ge \sum_{t=\hat{t}}^{L_j} (\Delta_{jt} - F_{jt})$$
$$\text{or} \quad \sum_{t=f}^{\hat{t}-1} (\Delta_{jt} - F_{jt}) \ge 0.$$

By the last inequality and property 2, we can see that the sum of savings in variable costs when opening the warehouse j in period f through $\hat{t}-1$ dominates the corresponding fixed costs. Thus the total costs can't be increased if warehouse is opened in period f through $\hat{t}-1$. Q. E. D.

If the free warehouse j is fixed open for $f \le t \le L_j$, then the sets K_{1t} and K_{2t} should be updated by $K_{1t} \cup \{j\}$ and $K_{2t} - \{j\}$ for $f \le t \le L_j$ respectively. And then L_j is updated by $f-1$.

Note that A_{jt}'s are nondecreasing from a node to its successor nodes since Δ_{jt}'s are nondecreasing. Thus NSS Ⅰ gives progressively better chances of fixing free warehouses open at the successor nodes.

By the similar reasoning as in NSS Ⅰ, we can get another node simplification step.

Simplification Step Ⅱ

At a node b & b tree, determine

$$B_{jf} = \min_{E_j \le t \le L_j} B_{jt} \quad \text{for } j \in \bigcup_{t=1}^{T} K_{2t}.$$

If $B_{jf} \le 0$, then there exists a best completion of this node such that $y_{jt}=0$ for $E_j \le t \le f$.

If the free warehouse j is fixed closed for $E_j \le t \le f$, then the sets K_{0t} and K_{2t} should be updated by $K_{0t} \cup \{j\}$ and $K_{2t} - \{j\}$ for $E_j \le t \le f$ respectively, and then E_j is updated by $f+1$. And as in the case of NSS Ⅰ, NSS Ⅱ gives progressively better chances of fixing free warehouses closed at successor nodes.

Noting that A_{jt} and B_{jt} depend on the sets K_{0t} and K_{1t}, the NSS Ⅰ and NSS Ⅱ are applied in a cyclic manner at a node until these simplifications can't be possible.

2. The Upper and Lower Bounds Employed

Once NSS Ⅰ & Ⅱ are applied completely, the upper and lower bounds are to be obtained.

The Upper Bound

The upper bounds to the problem P can be obtained at a nonterminal node by assigning the values 0 or 1 to all $y_{jt}, j\in K_{2t}$, $t=1,\cdots,T$ under the restriction that $y_{jt}\leqslant y_{jt+1}$, $t=1,\cdots,T$, $j=1,\cdots m$. Namely $\sum_t Z[K_{1t}\cup Y_t]$ can be an upper bound for any set $Y_t\subseteq K_{2t}$, $t=1,\cdots,T$ provided Y_t's satisfy the condition $Y_1\subseteq Y_2\subseteq,\cdots,\subseteq Y_T$. But in the process of applying the NSS I & II, we have already obtained the values of $Z[K_{1t}\cup Y_t]$ at a node for the following four types of $Y_t: K_{2t},\phi,K_{2t}-\{j\},\{j\}$ for $j\in K_{2t}$ respectively. So we have gotten the following upper bounds without further computation:

$$UB_1(j:t)=\sum_{p=1}^{E_j-1}Z[K_{1p}\cup K_{2p}]+\sum_{p=E_j}^{t-1}Z[K_{1p}\cup K_{2p}-\{j\}]+\sum_{p=t}^{T}Z[K_{1p}\cup K_{2p}]^{(*)}$$

$$\text{for } E_j\leqslant t\leqslant L_j+1,\ j\in\bigcup_{t=1}^{T}K_{2t}$$

$$UB_2(j:t)=\sum_{p=1}^{t}Z[K_{1p}]+\sum_{p=t+1}^{L_j}Z[K_{1p}\cup\{j\}]+\sum_{p=L_j+1}^{T}Z[K_{1p}]$$

$$\text{for } E_j-1\leqslant t\leqslant L_j,\ j\in\bigcup_{t=1}^{T}K_{2t}.$$

Thus the upper bound to the problem P can be set equal to the minimum value among these upper bounds. But we can say more.

Define $UB_1(j)$ and $UB_2(j)$ for $j\in\bigcup_{p=1}^{T}K_{2p}$ as follow:

$$UB_1(j)=\min_{E_j\leq t\leq L_j+1}UB_1(j:t)$$

$$UB_2(j)=\min_{E_j-1\leq t\leq L_j}UB_2(j:t).$$

Then the following property holds.

Property 3

$$UB_1(j)=UB_1(j:L_j+1)$$

$$UB_2(j)=UB_2(j:E_j-1).$$

Proof: We will show only that $UB_1(j)=UB_1(j:L_j+1)$. For the case of $UB_2(j)$, the procedure is basically same.

Noting that $j\in K_{0t}$ for $t<E_j$ and $j\in K_{1t}$ for $t>L_j$, it is easy to show that

$$UB_1(j:L_j+1)-UB_1(j:t)=\sum_{p=t}^{L_j}(\Delta_{jp}-F_{jp}),$$

$$\text{for } E_j\leqslant t\leqslant L_j.$$

But because NSS I has been applied, the right hand side of the above equation must be negative. Therefore

$UB_1(j:L_j+1)<UB_1(j:t)$ for $E_j\leqslant t\leqslant L_j$. Q.E.D.

Now, define UB_3, UB_4, C_{jt} and D_{jt} as follow:

$$UB_3=\sum_{p=1}^{T}Z[K_{1p}\cup K_{2p}]$$

$$UB_4=\sum_{p=1}^{T}Z[K_{1p}]$$

$$C_{jt}=\sum_{p=E_j}^{t}(F_{jp}-\Delta_{jp})$$

$$D_{jt}=\sum_{p=t}^{L_j}(\Omega_{jp}-F_{jp}).$$

Note that UB_3 and UB_4 are obviously special kinds of $UB_1(j:t)$ and $UB_2(j:t)$. Then

$$UB_1(j)-UB_3=\sum_{p=E_j}^{L_j}(\Delta_{jp}-F_{jp})<0 \text{ for } j\in\bigcup_{t=1}^{T}K_{2t}.$$

The last inequality holds because NSS I has been completely applied. Thus

$UB_1(j)=UB_3-C_{jL_j}$ and

$UB_1(j)<UB_3$.

Similarly,

$UB_2(j)=UB_4-D_{jE_j}$ and

$UB_2(j)<UB_4$.

Define UB_1 and UB_2 as follow:

$$UB_1=\min_{j\in\cup K_{2t}}UB_1(j)$$

$$UB_2=\min_{j\in\cup K_{2t}}UB_2(j).$$

Then thus far we have gotten the following facts.

Property 4

Determine

$$C_{kL_k}=\max_{j\in\cup K_{2t}}C_{jL_j}$$

$$D_{lL_l}=\max_{j\in\cup K_{2t}}D_{jL_j}.$$

Then $UB_1=UB_1(k)$ and $UB_2=UB_2(l)$.

Now we can obtain the upper bound UB to the problem P.

$$UB=\min(UB_1,UB_2)$$

(*) $\sum_{p=a}^{b}(\ )=0$, if $b<a$.

If UB is less than the incumbent value $\bar{Z}, \bar{Z}$ is updated by UB.

The Lower Bound

To find the lower bound LB on the optimal objective value of a CP at any nonterminal node, let's start with the assumed state where all warehouses in $\bigcup_{t=1}^{T}(K_{1t}\cup K_{2t})$ are all opened. The objective value at this state is UB_3. Now consider the physical meaning of C_{jt}. By the property 2, C_{jt} is an upper bound on the decrease in UB_3 with fixing the warehouse j closed in period E_j through t. Thus the objective value of the completion of CP, where each free warehouse j, $j\in \bigcup_{t=1}^{T} K_{2t}$ is closed in period E_j through r_j and open in all subsequent periods, can't be less than $UB_3 - \sum_{j\in\cup K_{2t}} C_{jr_j}$. Define

$$LB_1 = UB_3 - \sum_{j\in\cup K_{2t}} \max_{E_j\le r_j\le L_j} C_{jr_j}.$$

Then LB_1 is a lower bound on the optimal objective value of CP.

By the similar reasoning, if we define

$$LB_2 = UB_4 - \sum_{j\in\cup K_{2t}} \max_{E_j\le r_j\le L_j} D_{jr_j},$$

then LB_2 is also a lower bound.

Naturally, the eventual lower bound LB at a node is set to be the larger of these two, i.e.,

$$LB = \max(LB_1,\ LB_2).$$

If $LB > \bar{Z}$, the current node CP must be pruned.

3. The Node Simplification Steps Ⅲ&Ⅳ

NSS Ⅲ (or NSS Ⅳ) is a criterion for fixing free warehouses closed(or open) at a node at which $LB \leqslant \bar{Z}$.

Simplification Step Ⅲ

For each $k\in\bigcup_{t=1}^{T} K_{2t}$, define

$$LB_1^k = LB_1 + \max_{E_k\le r_k\le L_k} C_{kr_k}.$$

If $LB_1^k > \bar{Z}$, then the completions of the current CP better than the incumbent, if they exist, have $y_{kt}=0$ for $E_k \leqslant t \leqslant f$, where f is the smallest r for which $C_{kr} > LB_1^k - \bar{Z}$.

Proof: LB_1^k is a lower bound on the value of the completions of CP in which the warehouse k is open in all free periods. Thus if $LB_1^k > \bar{Z}$, the completions of CP better than the incumbent, if they exist, must have the warehouse k closed in period E_k through some period f, where $E_k \leqslant f \leqslant L_k$. But if the free warehouse k is closed in period E_k through p and open all subsequent periods, the objective value of the best completion of this new node must be greater than or equal to

$$UB_3 - C_{kp} - \sum_{\substack{j\in\cup K_{2t}\\ j\neq t}} \max_{E_j\le r_j\le L_j} C_{jr_j}. \qquad (9)$$

Note that the value of (9) is equal to $LB_1^k - C_{kp}$.

Thus if $LB_1^k - C_{kp} > \bar{Z}$, the objective value of the best completion of this new node should be greater than $\bar{Z}$. Therefore at the completions of CP better than the incumbent, if they exist, the warehouse k must be closed from period E_k to some period including at least the period f such that f is the smallest r for which $LB_1^k - C_{kr} < \bar{Z}$, i.e., $C_{kr} > LB_1^k - \bar{Z}$. Q.E.D.

By the similar reasoning, we can applying the following NSS Ⅳ.

Simplification Step Ⅳ

For each $k\in\bigcup_{t=1}^{T} K_{2t}$, define

$$LB_2^k = LB_2 + \max_{E_k\le r_k\le L_k} D_{kr_k}.$$

If $LB_2' > \bar{Z}$, then completions of the current CP better than the incumbent, if they exist, have $y_{kt}=1$ for $f \leqslant t \leqslant L_k$, where f is the largest r for which $D_{kr} > LB_2^k - \bar{Z}$.

If one or more free warehouses are fixed closed (or open) through the NSS Ⅲ (or NSS Ⅳ), we can again apply the NSS Ⅰ (or NSS Ⅱ) since the sets K_{0t} and K_{2t} (or K_{1t} and K_{2t}) are changed.

4. The Node Selection and Branching Rules

A node selection rule refers to the rule which determines the node to be considered next at the b & b tree. Among the several traditional rules, we can use the lowest lower bound(LLB) rule or the last-in first-out(LIFO) rule.

On the other hand, a branching rule is the rule which selects a free variable y_{jt}, $j \in K_{2t}$ to be constrained as 0 or 1 at the two successor nodes. Many branching rules can be considered for our problem as seen in Akinc and Khumawala[1] and Roodman and Schwarz[6]. Since the emphasis is put on the procedures of the *b & b* algorithm for solving the problem *P*, only one branch rule is presented here, which utilizes the available informations of C_{jt}'s. Note that C_{jt} represents the marginal savings associated with closing the warehouse j in period E_j through t. The following branching rule dictates that the variable which maximize the savings be selected.

Branching Rule

a) Select y_{kf} such that

$$C_{kf} = \max_{j,t} C_{jt}, \; j \in K_{2t}, \; t = 1, \cdots, T.$$

and

b) Create the two successor nodes:

1) a successor node in which warehouse j is fixed open in period f through L_k (i.e., k is added to K_{1t} and dropped from K_{2t} for $t = f, \cdots, L_k$, and L_k is set equal to $f-1$); and

2) another in which warehouse j is fixed closed in period E_k through f (i.e., k is added to K_{0t} and dropped from K_{2t} for $t = E_k, \cdots, f$, and then E_k is set equal to $f+1$).

Ⅳ. Discussion

Thus far, we have presented a new *b & b* algorithm to solve a model of a multi-period capacitated warehouse location problem. To reduce the size of the *b & b* tree, several node simplification steps have been devised. But in the process of of applying the algorithm, a large number of transshipment problems should be solved. For alleviating these difficulties, out-of-kilter method or primal-dual method could be used. It is known that out-of-kilter method is more efficient than primal-dual method on the average when the transshipment network is sparse. See Akinc and Khumawala[1]. Computational experience of the presented *b & b* algorithm is left for further researches.

References

1. Akinc. U. and Khumawala, B.M., "An Efficient Branch and Bound Algorithm for the Capacitated Warehouse Location Problem", *Management Science*, Vol. 23, No. 6, pp. 585—594(1977).
2. Choi, W.J., "Optimal Location of Warehouses in a Two-stage Distribution System", *unpublished M.S. Thesis*, K.A.I.S. (1979).
3. El-Shaieb, A.M., "A New Algorithm for Locating Sources Among Destinations", *Management Science*, Vol. 20, No. 2, pp. 221—233 (1973).
4. Eschenbach, T. and Carlson, R.C., "The Capacitated Multi-Period Location-Allocation Problem", *"Technical Report Sol* 75—27, Department of Operations Research, Stanford Univ. (1975).
5. Geoffrion, A.M., "A Guide to Computer Assisted Methods for Distribution System Planning", *Sloan Management Review*, Vol. 16, No. 2, pp. 17—41(1975).
6. Roodman, G.M. and Schwarz, L.B., 'Optimal and Heuristic Facility Phase-Out Strategies", *A.I.I.E. Transactions*, Vol. 7, No. 2, pp. 177—184(1975).
7. Roodman, G.M. and Schwarz, L.B., "Extension of the Multi-Period Facility Phase-Out Model", *A.I.I.E. Trans.*, Vol. 9, No. 1, pp. 103—107(1977).
8. Sweeney, D.J. and Tatham, R.L., "An Improved Long-Run Model for Multiple Warehouse Location", *Management Science*, Vol. 22, No. 7, pp. 748—758(1976).