On the Bochner-Herglotz-Weil-Raikov Theorem

Je. Hai-Gon
Dept. of Mathematics
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〈**Abstract**〉

In Bochner-Herglotz-Weil-Raikov Theorem, we may drop the assumptions of semisimplicity and selfadjointness for a Banach algebra and extend this theorem to a Banach *-algebra.

Bochner-Herglotz-Weil-Raikov 정리에 관하여

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〈요 약〉

Bochner-Herglotz-Weil-Raikov 정리에서 가정한 semisimple, selfadjoint의 조건없이 위정리를 Banach *-algebra에서 증명하였음.

I. Introduction

The well known theorem of Bochner-Herglotz-Weil-Raikov states that a linear functional f defined on a semisimple, selfadjoint, commutative Banach algebra A is positive and extendable if and only if there exists a finite positive Baire measure u on the maximal ideal space of A such that $f(x)=\int \hat{x}du$ for every x in A.

It is the puppose of this paper to extend above theorem to a Banach *-algebra without the assumptions of semisimplicity and selfadjointness for A.

A modified form of the Bochner-Herglotz-Weil-Raikov Theorem is stated in Theorem 3.1.

In case when A is a Banach algebra the above conditions may be dropped.

By the same argument as in pp. 98—99 of [3], it then comes out that the Plancherel Theorem can also be modified in a similar fashion as will be remarked at the end of this paper.

II. Notations and Preliminaries

Let A be a *-algebra; i.e., A is an algebra over the field of complex numbers with an involution, that is, a mapping : $x \to x^*$ of A onto A such that $(x+y)^*=x^*+y^*$, $(\alpha x)^*=\bar{\alpha}x^*$, $(xy)^*=y^*x^*$, $(x^*)^*=x$ for all x and y in A and complex numbers α. An element $x \in A$ is said to be selfadjoint if $\hat{x}^*=\bar{\hat{x}}$. If B is a subset of A we denote by B^* the set $\{x^* : x \in B\}$. A commutative Banach algebra A is said to be selfadjoint if for every x in A there exists a unique y in A such that $\hat{y}^*=\bar{\hat{x}}$. A linear functional f on a *-algebra A is said to be positive if $f(x^*x) \geq 0$ for all x in A. A positive linear functional f on a *-algebra A is said to be hermitian if $f(x^*)=\overline{f(x)}$ for all x in A. If f is any positive linear functional on A, then $f(x^*y)=\overline{f(y^*x)}$ and $|f(x^*y)|^2\leq f(x^*x)\cdot f(y^*y)$ for all x and y in A.

If A has an identity e, we can take $y=e$ and obtain $f(x^*)=\overline{f(x)}$ and $f(x)\leq Kf(x^*x)$, where $K=f(e)$. A positive linear functional which satisfies these extra conditions is called extendable.

A necessary and sufficient condition that a positive linear functional f on a *-algebra A without identity can be extended so as to remain positive when an identity is added to A is that f be extendable in the above sense.

For a fixed positive functional f defined on a dense ideal A_0 of an algebra A, an element $p\in A_0$ is said to be positive definite if the functional f_p defined on A by $f_p(x)=f(px)$ is positive and extendable.

We denote the space of all maximal ideals of an algebra A by M. If $M_0=M_0^*$ for any M_0 in M, then M_0 is said to be a symmetric maximal ideal of A.

Let Δ be the space of all continuous non-zero homomorphisms of an algebra A onto the complex numbers. For every x in A, the function $\hat{x}$ on Δ is defined by $\hat{x}(h)=h(x)$, $h\in\Delta$. We denote by $\hat{A}$ the algebra of all such functions $\hat{x}$ on Δ.

Any homomorphism of A into the algebra $L(X)$ is called a representation of A on X. Among the representations of an algebra, there is the so-called left (right) regular representation on the linear space A obtained by taking for each a in A the linear transformation T_a defined by $T_ax=ax$ $(T_ax=xa)$ for x in A.

The representation T_a is a *-representation if $L(X)$ has an involution and it is true that $T_{a^*}=(T_a)^*$.

For the proofs of the following lemmas which will be used, see PP. 197 of [1].

Lemma 2.1. Let f be a positive linear functional on a Banach *-algebra A. Then, for a, $b\in A$, $|f_b(a)|\leq f(b^*b)\ r(a^*a)^{1/2}$, where f_b is a linear functional defined by $f_b(a)=f(b^*ab)$ ($a\in A$) and $r(a)$ is a spectral radius of a.

Lemma 2.2. Let f be a positive linear functional on a Banach *-algebra A and let $b\in A$. Then f_b is continuous.

Lemma 2.3. Let A have a unit. Then every positive linear functional on A is continuous.

For convenience, we state here theorems of Bochner-Herglotz-Weil-Raikove and Plancherel, for the proof of which the reader is reffered to PP. 98—99 of [3].

Theorem 2.1 (Bochner-Herglotz-Weil-Raikov) If A is a semisimple, selfadjoint, commutative Banach algebra, then a linear functional f on A is positive and extendable if and only it there exists a finite positive Baire measure u on M such that $f(x)=\int\hat{x}\ du$ for every x in A.

Theorem 2.2 (Plancherel) Let A be a semisimple, selfadjoint, commutative Banach algebra and let f be a positive functional defined on dense ideal A_0 in A. Then there is a unique Baire measure u on M such that $p\in L^1(u)$ and $f(px)=\int\hat{x}\hat{p}du$ whenever p is positive definite with respect to f.

III. Bochner-Herglotz-Weil-Raikov Theorem on a complex commutative Banach *-algebra

Bochner-Herglotz-Weil-Raikov Theorem have assumed that A is a semisimple, selfadjoint commutative Banach algebra, that is, have assumed that $\hat{x}^*=\bar{\hat{x}}$. But the hypothesis that $\hat{x}^*=\bar{\hat{x}}$ can be dropped. We shall start with any commutative Banach *-algebra.

Let f be a positive linear functional on a commutative Banach *-algebra A. The element x in A such that $f(x^*x)=0$ form a left ideal I_f in A. If x is an element of A we denote by x_f the coset of $A/I_f=H'$ which contains x

and we define by

$$(x_f, y_f)=f(y^*x)$$

an inner product on H'. Thus H' becomes a pre-Hilbert space. Let H be the Hilbert space which is the completion of H'. If $a \in A$ we denote by $U : a \to U_a$ the left regular representation of A on H'.

Lemma 3.1. Let A be a commutative Banach $*$-algebra and f a positive functional of A. Then for any a in A U_a is bounded with respect to the f inner product norm and $a \to U_a$ is a $*$-representation of A on the Hilbert completion of H'.

Proof. Let $\|x_f\|$ be the f inner product norm in H'. By above lemmas for any a in A and x_f in H'. $\|U_a x_f\|^2=(U_a x_f, U_a x_f)=f((ax)^*ax)=f(x^*a^*ax)=f_x(a^*a)\leq f(x^*x)r(a^*a)$. Hence $\|U_a x_f\| \leq r(a^*a)^{1/2}\|x_f\|$. Thus U_a is bounded and in this case U_a can be uniquely extended to the Hilbert space which is the completion of H'.

Morever,

$$U_{a+b}x_f=((a+b)x)=(ax)+(bx)=U_a x_f+U_b x_f,$$
$$U_{ab}x_f=(abx)=U_a(bx)=U_a U_b x_f,$$
$$(U_a x_f, y_f)=f(y^*ax)=f((a^*y)^*x)=(x_f, U_a y_f).$$

This implies that $a \to U_a$ is a $*$-representation of A. ▮

The set $B=\{U_a : a \in A\}$ is a C*-algebra under the involution $U_a \to U_a^*$. We denote the space of all complex valued non-zero homomorphisms on B by Δ_1.

Lemma 2.2. Let h' be an element of Δ_1 and h a functional such that $h(a)=h'(U_a)$. Then h is an element of Δ and $\{h : h' \in \Delta_1,\ h(a)=h'(U_a)\}$ is a closed subset of Δ in the weak topology.

Proof. Let h' be an element of Δ_1 and h a functional such that $h(a)=h'(U_a)$. Since $U : a \to U_a$ is a $*$-represectation of A, $h'U$ is a homomorphism of A into the complex numbers and $h(a)=h'(U_a)=h'(U(a))=h'U(a)$, h is an element of Δ. We designate $h'U$ by $U^*\cdot h'$. If $h' \neq h''$, then $U^*h' \neq U^*h''$. Thus U^* is a 1−1 mapping of Δ_1 into Δ. The topology of Δ_1 is the weak topology defined by the algebra of functions $\widehat{U(a)}$. But $\widehat{U(a)}(h')=h'(U(a))=(U^*h')(a)=\hat{a}(U^*h')$ and since the function $\hat{a}$ define the topology of Δ, the mapping U^* is a homeomorphism.

Now let h be any homomorphism of Δ in the closure of $U^*(\Delta_1)$, that is, for $x_1 \cdots x_n$, there exists $h' \in \Delta_1$ such that $|h(x_i)-h'(U(x_i))|<\varepsilon$, $i=1 \ldots n$. This implies that, if $U(x_1)=U(x_2)$, then $h(x_1)=h(x_2)$ so that the functional h'' defined by $h''(U(x))=h(x)$ is single valued on B. Thus h'' is a homomorphism of B onto the complex numbers and $h(x)=h''(U(x))$, i.e. $h=U^*h''$. Hence $U^*(\Delta_1)$ is closed in Δ. ▮

Since B is a C*-algebra, B is symmetric and the maximal ideal space M_f of B is thus identifiable with a closed subset of Δ. The Bochner and Plancherel theorems are now confined to this subset M_f, which consists only of selfadjoint ideals and therefore renders unnecessary the assumption made above that $\hat{x}^*=\bar{\hat{x}}$ on the whole of Δ.

Theorem 3.1. Let A be a commutative Banach $*$-algebra. Then a linear functional f on A is positive and extendable if and only if there exists a finite positive Baire measure u on M_f such that $f(x)=\int \hat{x}\, du$ for every x in A, where $\hat{x}$ is restricted on M_f.

Proof. Since M_f is the space of all selfadjoint ideals of A, for any x in A and h in M_f $\overline{\hat{x}(h)}=\hat{x}^*(h)$. If $f(x)=\int \hat{x}\, du$ where u is a finite positive Baire measure on M_f, then

$$f(x^*x)=\int |\hat{x}|^2 du \geq 0,$$
$$f(x^*)=\int \hat{x}^* du=(\overline{\int \hat{x} du})=\overline{f(x)},$$
$$|f(x)|^2=|\int \hat{x} du|^2 \leq (\int |\hat{x}|^2 du)\cdot(\int 1\, du)=Kf(x^*x)$$

that is, f is positive and extendable. Conversely, if f is positive and extendable and if we denote by $\|x_f\|$ and $\|\hat{x}\|_1$ the f inner product norm and uniform norm of $C(M_f)$ respectively, then f is continuous and $|f(x)| \leq K\|x_f\| \leq B\|\hat{x}\|_1$. Since $\hat{A}_f=\{\hat{x} : h(x)=\hat{x}(h),\ h \in M_f\}$ is dense in

$C(M_f)$, by Stone-Weierstrass Theorem, the bounded linear functional I_f defined on $\hat{A}_f$ by $I_f(\hat{x})=f(x)$ can be extended in a unique way to $C(M_f)$. If $f'\in C(M_f)$ and $f'\geq 0$, then $f'^{1/2}$ can be uniformly approximated by functions $\hat{x}\in\hat{A}_f$ and hence f' can be uniformly approximated by function $|\hat{x}|^2$. Since $I_f(|\hat{x}|^2)=f(x^*x)\geq 0$ and $I_f(|\hat{x}|^2)$ approximates $I_f(f')$, it follows that $I_f(f')\geq 0$. That is, I_f is a bounded integral. *If* u is related measure, we have desired result $f(x)=\int\hat{x}du$ for all x in A.

Remark. Theorem 2.2 can be extended to the Banach *-algebra.

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