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Weakly Completely Continuous Functions

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〈Abstract〉

The authors introduce a new class of functions called weakly completely continuous and investigate the relation among the concepts of completely continuous, weakly completely continuous and faintly continuous functions.

약 완전 연속함수에 관하여

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〈요 약〉

우리는 약 완전 연속함수를 도입하여 완전 연속함수, 약 완전 연속함수 및 faintly 연속함수 사이의 관계를 연구한다.

I. Introduction

In 1974, Arya and Gupta[1] introduced the concept of completely continuous functions in topological spaces and showed that complete continuity is between continuity and strong continuity due to Levine[5]. The investigation of complete continuity was continued by the second author[9] of the present note. Recently, Reilly and Vamanamurthy[13] have improved upon several results established in[1] and [9]. The purpose of the present note is to introduce a new class of functions, called weakly completely continuous functions, as a generalization of completely continuous functions.

Throughout this note spaces will always mean topological spaces and $f: X \to Y$ denotes a function of a space X into a space Y. A subset S of a space X is said to be ***regular open***(resp. ***regular closed***) if $\mathrm{Int}(\mathrm{Cl}(S))=S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S))=S$), where $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$ denote the closure of S and the interior of S, respectively. A subset S of a space X is called ***θ-open***(resp. ***δ-open***) if for each $x \in S$, there exists an open set U such that $x \in U \subset \mathrm{Cl}(U) \subset S$(resp. $x \in U \subset \mathrm{Int}(\mathrm{Cl}(U)) \subset S$). The complement of a θ-open (resp. δ-open) set is called ***θ-closed***(resp. ***δ-closed***)[18].

II. Weakly Completely Continuous Functions

Definition 2.1. A function $f: X \to Y$ is said to be ***completely continuous***[1] if for each open set V of Y, $f^{-1}(V)$ is regular open in X.

A function $f: X \to Y$ is said to be ***strongly continuous***[5] if for each subset A of X, $f(\mathrm{Cl}$

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$(A))\subset f(A)$. It was shown in[1] that strong continuity implies complete continuity and complete continuity implies continuity and also the converses are not true in general.

Definition 2.2. A function $f: X \to Y$ is said to be ***weakly completely continuous*** (briefly WCC) if for each θ-open set V of Y, $f^{-1}(V)$ is regular open in X.

Theorem 2.3. A function $f: X \to Y$ is WCC if and only if for each θ-closed set F of Y, $f^{-1}(F)$ is regular closed in X.

Theorem 2.4. If $f: X \to Y$ is WCC, then the following equivalent properties hold:

(a) For each $x \in X$ and each θ-open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$.

(b) For each $x \in X$ and each θ-open set V containing $f(x)$, there exists a δ-open set U containing x such that $f(U) \subset V$.

(c) For any θ-open set V of Y, $f^{-1}(V)$ is δ-open in X.

(d) For any θ-closed set F of Y, $f^{-1}(F)$ is δ-closed in X.

Proof. It is obvious that weak complete continuity implies (a). Since a δ-open set is the union of regular open sets, it follows that (a) implies (b) and (c) implies (a). Since the union of δ-open sets is δ-open, we observe that (b) implies (c). It is obvious that (c) and (d) are equivalent.

Remark 2.5. In Theorem 2.4, (a) does not necessarily imply WCC. In Example 3.5(below), since X is regular, "open," "δ-open" and "θ-open" are equivalent and hence f satisfies (a) but it is not WCC.

Lemma 2.6. Let A be either dense or open in a space X. If U is a regular open set of X, then $A \cap U$ is regular open in the subspace A.

Proof. This follows from [17, Lemma] and [8, Theorem 4].

Theorem 2.7. If $f: X \to Y$ is WCC and A is either open or dense in a space X, then the restriction $f|A: A \to Y$ is WCC.

Proof. Let V be a θ-open set of Y. Then $f^{-1}(V)$ is regular open in X. It follows from Lemma 2.6 that $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is regular open in the subspace A. Therefore, $f|A$ is WCC.

Remark 2.8. Let $f: X \to Y$ be WCC and A, B subsets of X. Then, the restriction $f|A: A \to f(A)$ need not be WCC. Moreover, $f|A \cup B: A \cup B \to f(A \cup B)$ is not always WCC even if $f|A: A \to f(A)$, $f|B: B \to f(B)$ and f are all WCC.

Example 2.9. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\ \{a, b, c\}, X\}$. Let $A = \{a, b\}$ and $B = \{c\}$. Then, the identity function $f: (X, \tau) \to (X, \sigma)$, $f|A: A \to f(A)$ and $f|B: B \to f(B)$ are WCC. However, $f|A \cup B: A \cup B \to f(A \cup B)$ is not WCC.

Remark 2.10. There exists a WCC function under which the inverse image of a regular open set is not always regular open, as Example 3.6(below) shows. Therefore, we can conjecture that the composition of WCC functions is not always WCC. However, the present authors don't have the counterexample.

We shall obtain some conditions for the composiion of two functions to be WCC.

Definition 2.11. A function $f: X \to Y$ is said to be ***θ-continuous***[3] if for each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$.

It will be shown below that θ-continuity and weak complete continuity are independent of each other.

Lemma 2.12. (Jankovic[4]). If $f: X \to Y$ is θ-continous and V is θ-closed in Y then $f^{-1}(V)$ is θ-closed in X.

Theorem 2.13. If $f: X \to Y$ is WCC and $g: Y \to Z$ is θ-continous, then $g \circ f: X \to Z$ is WCC.

Proof. Let W be a θ-closed set of Z. By Lemma 2.12, $g^{-1}(W)$ is θ-closed in Y. Since f is WCC, by Theorem 2.3 $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is regular closed in X. It follows from

Theorem 2.3 that $g\circ f$ is WCC.

Definition 2.14. A function $f: X\to Y$ is called an *R-map*[2] if for each regular open set V of Y, $f^{-1}(V)$ is regular open in X.

It was shown in[2] that R-map and continuity are independent of each other. Every completely continuous function is an R-map but the converse is not true because complete continuity is strictly stronger than continuity [1, Example 6.2]. It will be shown below that WCC and R-map are independent of each other.

Theorem 2.15. If $f: X\to Y$ is an R-map and $g: Y\to Z$ is WCC, then $g\circ f: X\to Z$ is WCC.

Proof. Let W be a θ-open set of Z. Then $g^{-1}(W)$ is regular open in Y and hence $(g\circ f)^{-1}(W)=f^{-1}(g^{-1}(W))$ is regular open in X.

A space X is said to be ***extremally disconnected*** if the closure of each open set of X is open in X. A space X is said to be ***almost-regular***[14] if for each regular closed set F and each point $x\in X-F$, there exist disjoint open sets U and V such that $x\in U$ and $F\subset V$. It was shown in[14] that almost-regularity is implied by regularity and independent of semi-regularity, and also an almost-regular and semi-regular space is regular. Every open set of a regular space is θ-open. However, a θ-open set in a regular space is not necessarily regular open, as the following example shows.

Example 2.16. Let X be the real numbers with the usual topology and $A=(0,1)\cup(1,2)$. Then, A is open and hence θ-open in X, but it is not regular open in X.

Long and Herrington showed that a regular open set of an almost-regular space is θ-open [7, Theorem 3). However, a regular open set of a semi-regular space is not necessarily θ-open, as the following example shows.

Example 2.17. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then (X,τ) is semi-regular and $\{a\}$ is regular open in (X,τ), but it is not θ-open in (X,τ).

Lemma 2.18. If a space X is extremally disconnected, every regular open set of X is θ-open.

Proof. Let X be extremally disconnected and V regular open in X. Then, we have $V=\mathrm{Int}(\mathrm{Cl}(V))=\mathrm{Cl}(V)$ and hence V is open closed in X. Therefore, V is θ-open.

Theorem 2.19. Let Y be either extremally disconnected or almost-regular. If $f: X\to Y$ and $g: Y\to Z$ are WCC, then $g\circ f$ is WCC.

Proof. Let W be a θ-open set of Z. Then $g^{-1}(W)$ is regular open in Y and hence $g^{-1}(W)$ is θ-open in Y by Lemma 2.18 or [7, Theorem 3]. Therefore, $(g\circ f)^{-1}(W)=f^{-1}(g^{-1}(W))$ is regular open in X. This shows that $g\circ f$ is WCC.

A space X is said to be ***weakly-Hausdorff***[16] if each point of X is the intersection of regular closed sets of X.

Theorem 2.20. Let $f: X\to Y$ be a WCC injection. If Y is Hausdorff, then X is weakly-Hausdorff.

Proof. Let $x\in X$. Then $\{f(x)\}$ is a compact set of a Hausdorff space Y and hence $\{f(x)\}$ is θ-closed in Y [4, Theorem 4.3]. Since f is WCC and injective, $\{x\}=f^{-1}(f(x))$ is regular closed in X by Theorem 2.3. Therefore, X is weakly-Hausdorff.

Theorem 2.21. Let $f: X\to Y$ be a function and $g: X\to Y$ given by $g(x)=(x,f(x))$ for each $x\in X$, be the graph function. If g is WCC, then f is WCC.

Proof. Let V be a θ-open set of Y. It follows from Theorem 5 of[7] that $X\times V$ is a θ-open set of $X\times Y$. Since g is WCC, $g^{-1}(X\times V)$ is regular open in X. However, by a simple calculation we obtain $g^{-1}(X\times V)=f^{-1}(V)$. Therefore, f is WCC.

Ⅲ. Comparisons

Definition 3.1. A function $f: X\to Y$ is said to be ***faintly-continuous***[7] if for each $x\in X$ and each θ-open set V containing $f(x)$, there exists an open set U containing x such that $f(U)\subset V$. In[12] Prasad, Chae and Singh called faintly-continous functions weakly θ-continuous.

Theorem 3.2. For a function the following implications hold: complete continuity→weak complete continuity→faint-continuity.

Proof. Since every θ-open set is open, the first implication holds. The second implication follows from the result that a function is faintly-continuous if and only if the inverse image of each θ-open set is an open set [7, Theorem 9].

Definition 3.3. A function $f: X\to Y$ is said to be *δ-continuous*[10] (resp. *almost-continuous*[15], *weakly-continuous*[6)] if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Int}(\mathrm{Cl}(U))\subset \mathrm{Int}(\mathrm{Cl}(V))$ (resp. $f(U)\subset \mathrm{Int}(\mathrm{Cl}(V))$, $f(U)\subset \mathrm{Cl}(V)$).

Remark 3.4. For a function the following implications hold, but none of these implications is reversible[11,7].

(1) completely continuous→R-map→δ-continuous→almost-continuous.

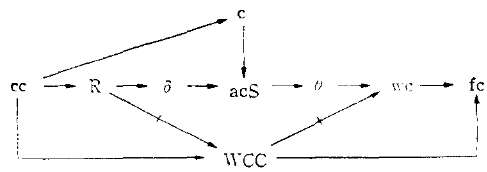
(2) continuous→almost-continuous→ θ-continuous→weakly-continuous → faintly-continuous.

We shall show that weak complete continuity is independent of each one of R-map, δ-continuity, continuity, almost-continuity, θ-continuity and weak-continuity.

Example 3.5. Let X be the real numbers with the usual topology and $f: X\to Y$ the identity function. Then f is continuous and also an R-map, but it is not WCC. Let $A=(0,1)\cup(1,2)$, then A is θ-open but it is not regular open in X.

Example 3.6. Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset, \{c\}, \{a,d\}, \{a,c,d\}, X\}$ and $\sigma=\{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$. Define a function $f: (X,\tau)\to(X,\sigma)$ as follows: $f(a)=f(b)=f(c)=a$ and $f(d)=b$. Then f is WCC but it is not weakly-continuous.

By Theorem 3.2, Remark 3.4 and Examples 3.5 and 3.6, we have the following diagram:



where "$A \not\to B$" represents that A does not always imply B, moreover, cc=completely continuous, R=R-map, δ=δ-continuous, c=continuous, acS=almost-continuous, θ=θ-continuous, wc=weakly-continuous, and fc=faintly-continuous.

In Theorem 11 of[7], Long and Herrington showed that if $f: X\to Y$ is faintly continuous and Y is almost-regular then f is almost-continuous. Similarly to the above theorem, we have

Theorem 3.7. Let Y be either almost-regular or extremally disconnected. If $f: X\to Y$ is WCC, then it is an R-map.

Proof. Let V be a regular open set of Y. It follows from Lemma 2.18 or [7, Theorem 3] that V is a θ-open set of Y. Since f is WCC, $f^{-1}(V)$ is regular open in X. This shows that f is an R-map.

Theorem 3.8. If $f: X\to Y$ is WCC and Y is a regular space, then f is completely continuons.

Proof. Let V be an open set of Y. Since Y is regular, V is θ-open in Y and hence $f^{-1}(V)$ is regular open in X. Therefore, f is completely continuous.

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