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A Note on a Fixed Point of a Nonexpansive Mapping in a Nonconvex Set and a Fixed Point Criterion for Compact T_2-Space

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〈Abstract〉

In this note I will prove a theorem concerning the existance of a fixed point of a nonexpansive mapping on certain class of a convex set and prove a fixed point theorem for compact T_2-space.

Nonconvex Set상의 Nonexpansive Mapping의 부동점과 Compact T_2-Space에 관한 부동점에 관한 연구

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〈요 약〉

본논문에서는 Nonconvex Set상의 Nonexpansive Mapping에 관한 부동점의 존재에 대하여 논하였고 다음에는 Compact T_2-Space에 대한 부동점에 관하여 고찰하였다.

I. On a Fixed point of a Nonexpansive Mapping in a Nonconvex set.

We let S be a subset of a Banch space E and $F=\{f_\alpha\}_{\alpha\in S}$ be a family of functions form $[0,1]$ into S, having the property that for each $\alpha\in S$ we have $f_\alpha(1)=\alpha$.

Definition 1. A family F is said to be contractive if there exists a function $\phi:(0,1)\to(0,1)$ such that for all α and β in S and for all t in $(0,1)$ we have

$$\|f_\alpha(t)-f_\beta(t)\|\leqq\phi(t)\|\alpha-\beta\|$$

Definition 2. A family F is said to be jointly continuous provided that if $t\to t_0$ in $[0,1]$ and $\alpha\to\alpha_0$ in S then $f_\alpha(t)\to f_{\alpha_0}(t_0)$ in S.

Theorem 1. Suppose S is a compact subset of a Banach space E, and suppose there exists a contractive, jointly continuous family F of functions associated with S as described above. Then any nonoxpansive self-mapping T of S has a fixed point in S.

Proof. For each $n=1,2,3,\cdots$, let $K_n=n/(n+1)$, and let T_n: $S\to S$ be defined by $T_nx=f_{Tx}(K_n)$ for all $x\in S$. Since $T(S)\subset S$ and $0<K_n<1$, we have that each T_n is well-defined and maps S into S. Furthermore, for each n we have, for all x,y in S,

$$\|T_nx-T_ny\|=\|f_{Tx}(K_n)-f_{Ty}(K_n)\|\leqq\phi(K_n)\|T_x-T_y\|\leqq\phi(K_n)\|x-y\|,$$

so that, for each n, T_n is a contraction mapping on S. As a compact (hence closed) subset of the Banach space E, S is a complete metric space. Therefore each T_n has a unique fixed point $x_n\in S$. Since S is compact, there is a subsequence $\{x_{nj}\}$ of $\{x_n\}$ such that $x_{nj}\to$some $x\in S$.

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Since $T_{nj}x_{nj}=x_{nj}$ we have $T_{nj}x_{nj}\to X$. But T is continuous (since nonexpansive), and so $T_{Xnj}\to T_X$ the joint continuity now yields

$$T_{nj}X_{nj}=f_{TXnj}(K_{nj})\to f_{Tx}(1)=T_X$$

It follows that $T_{X=X}$, since E is Hausdorff.

Ⅱ. A Fixed point Critarion for Compact T_2-Space.

We let x be a separated uniform space (see 3 for terminology and notaion). A self-mapping of X is any function from X into itself. (Note that we do not require any continuity of a self-mapping.) The symbol X^X will denote the space of all self-mappings of X equipped with the topology of uniform convergence on X.

Definition 3. When $\phi\in X^X$ and $F\subseteq X^X$, we will say that ϕ has fixed points modulo F provided that, for each $f\in F$, at least one of the functions $\phi_0 f$ and $f_0\phi$ has a fixed point.

We shall use the following Proposition 1, which is an easy consequence of the definitions.

Proposition 1. Let $\{x_\alpha: \alpha\in a\}$ be a net in a uniform space, X, which converges to some point $x_0\in X$. If $\{f_\alpha: \alpha\in a\}$ is a net in X^X which converges to a uniformly continuous f in X^X, then $\{f_\alpha(X_\alpha): \alpha\in a\}$ converges to $f(x_0)$ in X.

Theorem 2. Let X be a compact T_2-space, and let ϕ be a continuous self-mapping of X. If there is a net $\{f_\alpha: \alpha\in a\}$ in X^X such that (i) ϕ has fixed points modulo $\{f_\alpha: \alpha\in a\}$ and (ii) $\{f_\alpha: \alpha\in a\}$ converges to the identity function, id_X on X, then ϕ has a fixed point.

Proof. Condition (ii) requires that id_X be in the closure, in the topology on X^X, of the collection $F=\{f_\alpha: \alpha\in a\}$. If we let $F_R=\{f_\alpha\in F: f_{\alpha 0}\phi$ has a fixed point$\}$ and $F_L=\{f_\alpha\in F: \phi^0 f_\alpha$ has a fixed point$\}$, then, by (i), $F=F_R\cup F_L$. Thus, we can select a subnet $\{f_\beta: \beta\in B\}$ of F which converges to id_X and is entirely contained within F_R or F_L. In the first case, we define ϕ_β $X\to X$ for each $\beta\in B$ by $\phi_B(x)=f_B(\phi(x))$; in the second case, we define $\phi_B(x)=\phi(f_B(x))$. In either case, for each $\beta\in B$, there is and x_β in X such that $\phi_\beta(x_\beta)=x_\beta$. Since X is compact, we may assume that $\{x_\beta: \beta\in B\}$ converges to some x_0 in X. But $\{\phi: \beta\in B\}$ converges to ϕ in X^X, so aopplication of the proposition yields that $x_0=\lim x_\beta=\lim \phi_\beta (x_\beta)=\phi(x_0)$, and x_0 is a fixed point of ϕ.

References

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