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ASYMPTOTIC BEHAVIOR OF NUMERICAL SOLUTION FOR THE VISCOUS CAHN-HILLIARD EQUATION

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ABSTRACT. – Numerical solutions for the viscous Cahn-Hilliard equation are considered using Crank-Nicolson type finite difference method. The corresponding stability and error analysis of the scheme are shown. The decay speeds of the solution in H^1-norm are shown.
Keywords– Viscous Cahn-Hilliard equation, conservation of mass, decay property, nonlinear difference scheme.

1. Introduction

Consider the viscous Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} - \delta \frac{\partial^3 u}{\partial x^2 \partial t} + \alpha \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 \phi(u)}{\partial x^2}, \quad x \in \Omega, \quad 0 < t, \tag{1.1a}$$

with an initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.1b}$$

and boundary conditions

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \partial\Omega, \quad 0 < t. \tag{1.1c}$$

The constants δ, α and γ are positive, $\Omega = (0,1)$ with its boundary $\partial\Omega$. Here δ and α are coefficients of viscosity and gradient energy, respectively. The function $\phi(u) = \gamma u^3 - \beta^2 u$ is an intrinsic chemical potential and $u(x,t)$ is the concentration of one of two components of the system.

The equation (1.1) with $\delta > 0$ arises as a phenomenological continuum model for phase separation in glass and polymer systems where intermolecular friction forces may be expected to be of importance. See Novick-Cohen[12] for a derivation of the model and Novick-Cohen and Pego [13] for more physical motivation. The viscous Cahn-Hilliard equation, which is viewed as a singular limit of the phase field model of phase transition, has been studied by Bai, Elliott, Gardiner, Spence, and Stuart [2]. They have studied

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX the similarities and differences between the Cahn-Hilliard equation($\delta = 0$) and Allen-Cahn equation by using the viscous Cahn-Hilliard equation. Metastable pattern for the viscous Cahn-Hilliard equation has been studied by Reyna and Ward[14]. Using explicit energy calculations, Grinfeld and Novick-Cohen[11] have established a Morse decomposition of the stationary solutions of the viscous Cahn-Hilliard equation. Existence theory of the solution of (1.1) has been shown in Elliott and Stuart[9]. Choo and Chung[3] have investigated the exponential decay of the classical solutions of (1.1) and compared decay speeds of the viscous Cahn-Hilliard equation with that of the Cahn-Hilliard equation analytically.

Compared to numerical studies for the Cahn-Hilliard equation by Elliot and French[6]-[7], Elliot, French and Milner[8] with finite element methods and Furihata, Onda, and Mori[10], Sun[15], Choo and Chung[4], Choo, Chung and Kim [5] with finite difference methods, there is no numerical study for the viscous Cahn-Hilliard equation.

In this paper, a nonlinear difference scheme for (1.1) is considered which inherits mass conservation and dissipation property. We show numerically that the solutions of the corresponding nonlinear difference scheme have decay properties.

The outline of this paper is as follows. In section 2, a nonlinear conservative difference scheme for the viscous Cahn-Hilliard equation is introduced. In section 3, existence and error analysis of the scheme are shown. In section 4, we show the decay property of numerical solutions of the viscous Cahn-Hilliard equation. The speeds of the decay and energy dissipation property are shown.

2. Nonlinear finite difference scheme

Let $h = \frac{1}{M}$ be the uniform step size in the spatial direction for a positive integer M and $\Omega_h = \{x_i = ih | i = 0, \ldots, M\}$. Let $k = \frac{T}{N}$ denote the uniform step size in the temporal direction for any positive integer N. Denote $V_i^m = V(x_i, t_m)$ for $t_m = mk, m = 0, 1, \cdots$. For a function $V^m = (V_0^m, V_1^m, \cdots, V_M^m)$ defined on Ω_h, define the difference operators as

$$\nabla_+ V_i^m = \frac{V_{i+1}^m - V_i^m}{h}, \qquad \nabla_- V_i^m = \frac{V_i^m - V_{i-1}^m}{h},$$

$$\nabla^2 V_i^m = \begin{cases} \nabla_+(\nabla_- V_i^m), & \text{for } 1 \le i \le M-1, \\ 2\frac{V_1^m - V_0^m}{h^2}, & \text{for } i = 0, \\ 2\frac{V_{M-1}^m - V_M^m}{h^2}, & \text{for } i = M, \end{cases}$$

and

$$\nabla^4 V_i^m = \nabla^2(\nabla^2 V_i^m).$$

Define difference operators $V^{m+1/2}$ and $\partial_t V^m$, respectively, as

$$V_i^{m+1/2} = \frac{V_i^{m+1} + V_i^m}{2}, \quad \partial_t V_i^m = \frac{V_i^{m+1} - V_i^m}{k}.$$

Then the approximate solution U^m for (1.1) is defined as a solution of

$$\partial_t U_i^m - \delta \partial_t \nabla^2 U_i^m + \alpha \nabla^4 U_i^{m+1/2} = \nabla^2 \phi(U_i^{m+1/2}), \qquad 0 \le i \le M, \ 0 \le m, \qquad (2.1a)$$

with initial conditions

$$U_i^0 = u_0(x_i) \tag{2.1b}$$

and boundary conditions

$$\nabla_- U_0^0 = 0 = \nabla_+ U_M^0. \tag{2.1c}$$

We now introduce the discrete L^2 inner product and the corresponding discrete L^2-norm, respectively,.

$$(V,W)_h = h\{\frac{1}{2}(V_0W_0 + V_MW_M) + \sum_{i=1}^{M-1} V_iW_i\}, \quad \|V\|_h = (V,V)_h^{\frac{1}{2}}$$

for functions $V = (V_0, \dots, V_M)$ and $W = (W_0, \dots, W_M)$ defined on Ω_h. And $\|V\|_\infty = \max_{0\le i\le M}|V_i|$. Whenever there is no confusion, $(\cdot,\cdot)$ and $\|\cdot\|$ will denote $(\cdot,\cdot)_h$ and $\|\cdot\|_h$, respectively.

From summation by parts the following identities are obtained.

Lemma 2.1. *If V and W are functions defined on Ω_h with $\nabla_- V_0 = 0 = \nabla_+ V_M$ and $\nabla_- W_0 = 0 = \nabla_+ W_M$, then*

(1) $(\nabla^2 V, W) = -h\sum_{i=1}^{M}(\nabla_- V_i)(\nabla_- W_i)$.
(2) $(\nabla^2 V, W) = (V, \nabla^2 W)$.
(3) $(\nabla^4 V, W) = (\nabla^2 V, \nabla^2 W)$.

The following lemma can be verified by summation by parts and minimum eigenvalue of a symmetric matrix(see [1]).

Lemma 2.2. *Let M be any positive integer. Then*

(1) *If $V_0 = 0$, then* $\{2\sin\frac{\pi}{2(2M+1)}\}^2 \sum_{i=1}^{M} V_i^2 \le \sum_{i=1}^{M}(V_i - V_{i-1})^2$.
(2) *If $V_0 = V_{M+1} = 0$, then* $\{2\sin\frac{\pi}{2(M+1)}\}^2 \sum_{i=1}^{M} V_i^2 \le \sum_{i=1}^{M+1}(V_i - V_{i-1})^2$.

Then the following discrete Poincaré inequalities are obtained.

Lemma 2.3. *Let U be a discrete function defined on Ω_h. Then, for $0 < h \le 1$,*

$$\|U\|^2 \le 2\|\nabla_- U\|^2 + \frac{\pi^2}{2}\mathcal{M}^2,$$

where $\mathcal{M} = h\{\frac{1}{2}(U_0 + U_M) + \sum_{i=1}^{M-1} U_i\}$.

Proof. Replacing $V_i = U_i - U_0$ in (1) of Lemma 2.2, we obtain

$$\frac{1}{h^2}\left[\{2\sin\frac{\pi}{2(2M+1)}\}^2 h\sum_{i=1}^{M} U_i^2 + Mh\{2\sin\frac{\pi}{2(2M+1)}\}^2 U_0^2\right]$$
$$\le h\sum_{i=1}^{M}(\nabla_- U_i)^2 + \frac{2}{h}\{2\sin\frac{\pi}{2(2M+1)}\}^2\sum_{i=1}^{M} U_0 U_i$$
$$\le 2\|\nabla_- U\|^2 + \frac{1}{h}\{2\sin\frac{\pi}{2(2M+1)}\}^2(U_0^2 + \frac{1}{4}U_M^2)$$
$$+ \frac{1}{h^2}\{2\sin\frac{\pi}{2(2M+1)}\}^2 2(\frac{1}{4}U_0^2 + \mathcal{M}^2) - \frac{1}{h}\{2\sin\frac{\pi}{2(2M+1)}\}^2 U_0^2. \tag{2.2}$$

Thus it follows from (2.2) that

$$\{\frac{2\sin\frac{\pi h}{4+2h}}{h}\}^2\|U\|^2 \le 2\|\nabla_- U\|^2 + 2\{\frac{2\sin\frac{\pi h}{4+2h}}{h}\}^2\mathcal{M}^2.$$

Since $0 < \frac{\pi h}{4+2h} \le \frac{\pi}{6}$ and $1 \le \frac{2}{h}\sin\frac{\pi h}{4+2h} \le \frac{\pi}{2}$, we obtain the desired result. □

Lemma 2.4. *If V is a function defined on Ω_h with $\nabla_- V_0 = 0$, then*

(1) $\|\nabla_- V\| \le \frac{\sqrt{5}}{h}\|V\|$.
(2) $\|\nabla^2 V\| \le \frac{4}{h^2}\|V\|$.
(3) $\|\nabla_- V\| \le \|\nabla^2 V\|$.
(4) $\|\nabla_- V\|_\infty^2 \le \frac{5}{2}\|\nabla^2 V\|^2$.
(5) $\|V\|_\infty^2 \le 3\|V\|^2 + \|\nabla_- V\|^2$.

Proof. Using the definition of discrete norms, we obtain the discrete inverse inequalities (1) and (2). Since (3) comes directly from Lemma 2.2, we have only to prove the inequality (4) and (5).

Since $\nabla_- V_0 = 0$, adding and subtracting, we obtain

$$(\nabla_- V_i)^2 = \{(\nabla_- V_i)^2 - (\nabla_- V_{i-1})^2\} + \cdots + \{(\nabla_- V_1)^2 - (\nabla_- V_0)^2\}$$
$$= \sqrt{h}\nabla^2 V_{i-1}\sqrt{h}(\nabla_- V_i + \nabla_- V_{i-1}) + \cdots + \sqrt{h}\frac{1}{2}\nabla^2 V_0\sqrt{h}(\nabla_- V_1 + \nabla_- V_0)$$
$$\le \frac{1}{2}\|\nabla^2 V\|^2 + 2\|\nabla_- V\|^2.$$

Thus $\|\nabla_- V\|_\infty^2 \le \frac{1}{2}\|\nabla^2 V\|^2 + 2\|\nabla_- V\|^2$.

To prove the inequality (5), we can assume $V_0^2 \le V_i^2$ for all i without loss of generality. Since

$$V_i^2 = \sqrt{h}(V_i + V_{i-1})\sqrt{h}(\nabla_- V_i) + \cdots + \sqrt{h}(V_1 + V_0)\sqrt{h}(\nabla_- V_1) + V_0^2$$
$$\le 3\|V\|^2 + \|\nabla_- V\|^2,$$

we obtain the desired results. □

It is clear that the conservation of mass for (2.1) holds as for the classical solution.

Theorem 2.1. *Let U^n be the solution of (2.1). Then the conservation of mass holds. That is,*

$$h\{\frac{1}{2}(U_0^n + U_M^n) + \sum_{i=1}^{M-1} U_i^n\} = \mathcal{M}.$$

3. Convergence of approximate solution

In this section, we show existence and stability of solutions for the difference scheme (2.1). We also consider the error analysis for (2.1). The proof of the following lemma can be seen in Choo and Chung[4].

Lemma 3.1. *If U is a function defined in Ω_h and $\phi(U) = \gamma(U)^3 - \beta^2 U$, then*

$$-(\nabla^2\phi(U), U) \geq \beta^2(\nabla^2 U, U), \text{ and } -(\nabla^2\phi(U), U - \mathcal{M}) \geq \beta^2(\nabla^2 U, U).$$

We now prove the existence of solutions for (2.1) using the Brouwer fixed point theorem.

Theorem 3.1. *The solution U^n of the finite difference scheme (2.1) exists when $\alpha > 2\beta^2$.*

Proof. In order to prove this theorem by the induction, assume that $U^0, U^1, \ldots, U^n$ exist. Let g be a function defined by

$$g(W) = 2W - 2U^n - 2\delta\nabla^2 W + 2\delta\nabla^2 U^n + k\alpha\nabla^4 W - k\nabla^2\phi(W).$$

Then g is clearly continuous. Taking an inner product of $g(W)$ with W and applying Young's inequality, we have

$$\begin{aligned}(g(W), W) \geq\ & 2\|W\|^2 - 2\|U^n\|\|W\| - 2\delta(\nabla^2 W, W) - 2\delta\|\nabla^2 U^n\|\|W\| \\ & + k\alpha\|\nabla^2 W\|^2 - k(\nabla^2\phi(W), W).\end{aligned}$$

It follows from Lemma 3.1, Lemma 2.1 and Lemma 2.4 that

$$\begin{aligned}(g(W), W) \geq\ & \|W\|^2 - 2\delta(\nabla^2 W, W) + k(\alpha - 2\beta^2)\|\nabla^2 W\|^2 \\ & - 2\|U^n\|^2 - 2\delta^2\|\nabla^2 U^n\|^2.\end{aligned} \tag{3.1}$$

Since $(\nabla^2 W, W) = -h\sum_{i=1}^{M}(\nabla_- W_i)^2$ and $\alpha > 2\beta^2$, (3.1) becomes

$$(g(W), W) \geq \|W\|^2 - 2\|U^n\|^2 - 2\delta^2\|\nabla^2 U^n\|^2.$$

The Brouwer fixed point theorem implies the existence of W for $g(W) = 0$. Then $U^{n+1} = 2W - U^n$ satisfies (2.1). □

Remark 3.1. Since $-(\nabla^2\phi(W), W) \geq \beta^2(\nabla^2 W, W) \geq -\alpha\|\nabla^2 W\|^2 - \frac{\beta^4}{4\alpha}\|W\|^2$, the solution of (2.1) exists if $k \leq \max\{\frac{4\alpha}{\beta^4}, \frac{2\delta}{\beta^2}\}$.

Remark 3.2. Since $-(\nabla^2\phi(W), W) \geq \beta^2(\nabla^2 W, W) \geq -\beta^2 \frac{1}{\frac{1}{h^2}(2\sin\frac{\pi h}{2+2h})^2}\|\nabla^2 W\|^2$, the solution of (2.1) exists if $\alpha > \frac{\beta^2}{\pi^2}$ and $0 < h \leq h_0$ such that

$$\pi^2 - \frac{\pi^2\alpha - \beta^2}{2\alpha} \leq \frac{1}{h^2}(2\sin\frac{\pi h}{2+2h})^2.$$

We now show that the finite difference scheme (2.1) preserves dissipation property as the classical solution of (1.1) does.

Theorem 3.2. *Let $\alpha > 2\beta^2$ and U^n be the solution of (2.1). Then*

$$\|U^n\| - \delta(\nabla^2 U^n, U^n) \leq \|U^{n-1}\| - \delta(\nabla^2 U^{n-1}, U^{n-1}).$$

Proof. Forming an inner product between (2.1) and $U^{m+1/2}$ and applying Lemma 2.1 and Lemma 2.4, we obtain

$$\frac{1}{2}\partial_t\|U^m\|^2 - \frac{\delta}{2}\partial_t(\nabla^2 U^m, U^m) + (\alpha - 2\beta^2)\|\nabla^2 U^{m+1/2}\|^2 \leq 0. \tag{3.2}$$

Since the last term on the left is positive, the proof is completed. □

Remark 3.3. It follows from Theorem 3.2 that the scheme (2.1) is unconditionally stable and

$$\|U^n\| + \|\nabla_- U^n\|^2 \leq 4\|U^0\| - 4\delta(\nabla^2 U^0, U^0). \tag{3.3}$$

Let $u^n \equiv u(\cdot, t_n)$ and U^n be solutions for (1.1) and (2.1), respectively. Let $e^n = u^n - U^n$ be the error. Then we obtain the following error estimates for (2.1).

Theorem 3.3. *Assume $\alpha > 2\beta^2$. Then there exists a constant C such that*

$$\|e^n\| + \|\nabla_- e^n\| \leq C(k^2 + h^2).$$

Proof. Replacing $U^m = u^m - e^m$ in (2.1) and forming the inner product with $e^{m+1/2}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t\{\|e^m\|^2 - \delta(\nabla^2 e^m, e^m)\} + \alpha\|\nabla^2 e^{m+1/2}\|^2 \\
&= (u_t(t_{m+\frac{1}{2}}) - \delta\frac{\partial^2 u_t}{\partial x^2}(t_{m+\frac{1}{2}}) + \alpha\frac{\partial^4 u}{\partial x^4}(t_{m+\frac{1}{2}}) + O(h^2 + k^2), e^{m+1/2}) \\
&\quad - (\nabla^2\phi(U^{m+1/2}), e^{m+1/2}) \\
&\leq (\gamma\nabla^2(u^{m+1/2})^3 - \gamma\nabla^2(U^{m+1/2})^3 - \beta^2\nabla^2 e^{m+1/2}, e^{m+1/2}) \\
&\quad + \|e^{m+1/2}\|^2 + O(h^4 + k^4) \\
&\leq \gamma([(u^{m+1/2})^2 + u^{m+1/2}U^{m+1/2} + (U^{m+1/2})^2]e^{m+1/2}, \nabla^2 e^{m+1/2}) \\
&\quad - \beta^2(\nabla^2 e^{m+1/2}, e^{m+1/2}) + \|e^{m+1/2}\|^2 + O(h^4 + k^4).
\end{aligned}$$

It follows from (3.3) and Lemma 2.4 that there exists a constant C such that

$$\partial_t\{\|e^m\|^2 - \delta(\nabla^2 e^m, e^m)\} \leq C(\|e^{m+1}\|^2 + \|e^m\|^2) + C(h^4 + k^4).$$

Summing from $m = 0$ to $n-1$, we obtain

$$(1-Ck)\|e^n\|^2 - \delta(\nabla^2 e^n, e^n) \leq (1-Ck)\|e^0\|^2 - \delta(\nabla^2 e^0, e^0) + Ck\sum_{m=0}^{n-1}\|e^m\|^2 + Ck(h^4+k^4).$$

Applying Lemma 2.1 and the discrete Gronwall inequality with small k such that $1-Ck > 0$, we complete the proof. □

4. Asymptotic behavior of approximate solution

In this section, we show the decay property of solutions for the difference scheme (2.1).

Theorem 4.1. *Let U^n be the solutions of (2.1) and assume $\alpha > 2\beta^2$. Then we have*

$$\|U^n - \mathcal{M}\|^2 + \frac{\delta}{4}\|\nabla_- U^n\|^2 \leq \frac{\mathcal{C}_0 + \mathcal{C}_1}{1 + k\sum_{m=1}^{n}\frac{1}{\mathcal{C}_0 m + \mathcal{C}_1}}\{\|\nabla_- U^{1/2}\|^2 + \frac{1}{4}\|\nabla_-(U^1 - U^0)\|^2\}$$

where $\mathcal{C}_0 = (\frac{17}{2} + \frac{4}{\delta})k + \frac{4(1+\delta)}{\alpha - 2\beta^2}$ and $\mathcal{C}_1 = \frac{4(1+\delta)}{\alpha-2\beta^2}$.

proof. Forming the inner product between (2.1) and $U^m - \mathcal{M}$, it follows from Lemma 3.1, Lemma 2.1 and Lemma 2.4 that

$$\partial_t\{\|U^m - \mathcal{M}\|^2 - \delta(\nabla^2 U^m, U^m)\} + 2(\alpha - 2\beta^2)\|\nabla_- U^{m+1/2}\|^2 \leq 0. \tag{4.1}$$

Let $m^*(0 \leq m^* \leq m)$ be the number such that

$$\sum_{i=0}^{1}\|\nabla_- U^{m^*+i}\|^2 = \max_{0\leq j\leq m}\{\sum_{i=0}^{1}\|\nabla_- U^{j+i}\|^2\}. \tag{4.2}$$

Then we obtain for $m \geq 1$

$$\begin{aligned}\partial_t\Big\{&\sum_{j=0}^{m}\{\|U^j - \mathcal{M}\|^2 - \delta(\nabla^2 U^j, U^j)\}\\ &+ 2k(\alpha - 2\beta^2)\sum_{j=0}^{m-1}\sum_{i=0}^{1}\{\frac{1}{\delta}\|U^{j^*+i} - \mathcal{M}\|^2 - (\nabla^2 U^{j^*+i}, U^{j^*+i})\}\Big\}\\ &+ 2(\alpha - 2\beta^2)\|\nabla_- U^{m^*+1/2}\|^2 \leq 0.\end{aligned} \tag{4.3}$$

Adding $2(\alpha-2\beta^2)\{(\nabla_-U^{m^*+1},\nabla_-U^{m^*})-(\nabla_-U^{m^*+1},\nabla_-U^{m^*})\}$ to the left hand side of (4.3), we obtain

$$\partial_t\Big\{\tilde U^m+\sum_{j=0}^{m}\{\|U^m-\mathcal{M}\|^2-\delta(\nabla^2U^m,U^m)\}$$
$$+2k(\alpha-2\beta^2)\sum_{j=0}^{m-1}\sum_{i=0}^{1}\{\frac{1}{\delta}\|U^{j^*+i}-\mathcal{M}\|^2-(\nabla^2U^{j^*+i},U^{j^*+i})\}\Big\}$$
$$+2(\alpha-2\beta^2)\frac{1}{4}\|\nabla_-U^{m^*+1}-\nabla_-U^{m^*}\|^2\le 0 \tag{4.4}$$

where $\tilde U^m=2k(\alpha-2\beta^2)\sum_{j=0}^{m-1}(\nabla_-U^{j^*+1},\nabla_-U^{j^*})$ with $\tilde U^0=0$. It follows from (4.3) and (4.4) that

$$0\ge\partial_t\Big\{\tilde U^m+2\sum_{j=0}^{m}\{\|U^j-\mathcal{M}\|^2-\delta(\nabla^2U^j,U^j)\}$$
$$+4k(\alpha-2\beta^2)\sum_{j=0}^{m-1}\sum_{i=0}^{1}\{\frac{1}{\delta}\|U^{j^*+i}-\mathcal{M}\|^2-(\nabla^2U^{j^*+i},U^{j^*+i})\}\Big\}$$
$$+2(\alpha-2\beta^2)\{\|\nabla_-U^{m^*+1/2}\|^2+\frac{1}{4}\|\nabla_-U^{m^*+1}-\nabla_-U^{m^*}\|^2\}$$
$$\equiv\partial_tA(U^m)+B(U^m). \tag{4.5}$$

Note that for $0\le j\le m$,

$$(\alpha-2\beta^2)(\|\nabla_-U^{j+1}\|^2+\|\nabla_-U^j\|^2)\le B(U^m)\le B(U^{m+1}). \tag{4.6}$$

Since

$$A(U^m)=k(\alpha-2\beta^2)\sum_{j=0}^{m-1}\{\|\nabla_-U^{j^*+1}\|^2+2(\nabla_-U^{j^*+1},\nabla_-U^{j^*})+\|\nabla_-U^{j^*}\|^2\}$$
$$+k(\alpha-2\beta^2)\sum_{j=0}^{m-1}\sum_{i=0}^{1}\{-\|\nabla_-U^{j^*+i}\|^2-4(\nabla^2U^{j^*+i},U^{j^*+i})\}$$
$$+2\sum_{j=0}^{m}\{\|U^j-\mathcal{M}\|^2-\delta(\nabla^2U^j,U^j)\}+4k(\alpha-2\beta^2)\sum_{j=0}^{m-1}\sum_{i=0}^{1}\frac{1}{\delta}\|U^{j^*+i}-\mathcal{M}\|^2, \tag{4.7}$$

$$A(U^m)\ge 2\{\|U^m-\mathcal{M}\|^2-\delta(\nabla^2U^m,U^m)\}\ge 2\{\|U^m-\mathcal{M}\|^2+\frac{\delta}{4}\|\nabla_-U^m\|^2\}.$$

It follows from (4.6)–(4.7) and Lemma 2.1 that

$$\begin{aligned} A(U^{m+1}) &\le k(\alpha-2\beta^2)\sum_{j=0}^{m}\|\nabla_- U^{j^*+1/2}\|^2 + 4k(\alpha-2\beta^2)\sum_{j=0}^{m}\sum_{i=0}^{1}2\|\nabla_- U^{j^*+i}\|^2 \\ &+ 2\sum_{j=0}^{m+1}(2+\delta\cdot 2)\|\nabla_- U^{j}\|^2 + 4k(\alpha-2\beta^2)\sum_{j=0}^{m}\sum_{i=0}^{1}\frac{1}{\delta}\|\nabla_- U^{j^*+i}\|^2 \\ &\le \{\frac{k}{2}(m+1) + 8k(m+1) + \frac{4(1+\delta)}{\alpha-2\beta^2}(m+2) + \frac{4}{\delta}k(m+1)\}B(U^m) \\ &\equiv \{\mathcal{C}_0(m+1)+\mathcal{C}_1\}B(U^m) \end{aligned} \tag{4.8}$$

where $\mathcal{C}_0 = (\frac{17}{2}+\frac{4}{\delta})k + \frac{4(1+\delta)}{\alpha-2\beta^2}$ and $\mathcal{C}_1 = \frac{4(1+\delta)}{\alpha-2\beta^2}$. The inequality (4.5) implies

$$A(U^{m+1}) - A(U^m) + kB(U^m) \le 0,$$

and using (4.8), we obtain $A(U^m) \le \frac{\mathcal{C}_0 m+\mathcal{C}_1}{\mathcal{C}_0 m+\mathcal{C}_1+k}A(U^{m-1})$. This completes the proof. □

Remark 4.1. Replacing (4.1) by (3.2), we obtain from (4.5)

$$\partial_t A(U^m) + (\alpha-2\beta^2)\{\|\nabla^2 U^{m+1}\|^2 + \|\nabla^2 U^m\|^2\} \le 0.$$

Thus $\|\nabla^2 U^m\|^2 \to 0$ as $n\to\infty$.

Remark 4.2. Assume $\alpha > 2\beta^2$. Forming the inner product between (2.1) and $\partial_t U^m$, we obtain $\|\partial_t U^m\|^2 - \delta(\nabla^2\partial_t U^m, \partial_t U^m) + \frac{1}{2}\alpha\partial_t\|\nabla^2 U^m\|^2 = (\nabla^2\phi(U^{m+1/2}), \partial_t U^m)$. It follows from Lemmas 2.2–2.4 that for some $m_0 \ge 0$

$$\begin{aligned} &k\sum_{m=m_0}^{n-1}\{\|\partial_t U^m\|^2 - \delta(\nabla^2\partial_t U^m, \partial_t U^m)\} + \alpha\|\nabla^2 U^n\|^2 \\ &\le \alpha\|\nabla^2 U^{m_0}\|^2 + 2[\frac{180}{\delta h}\gamma(4+\pi^2\mathcal{M}^2+\frac{\pi^4}{4}\mathcal{M}^4)+\beta^4]k\sum_{m=m_0}^{n-1}\|\nabla^2 U^{m+1/2}\|^2. \end{aligned}$$

Since the right hand side in the above inequality is less than a constant independent of n, we obtain $\|\partial_t U^n\| - \delta(\nabla^2\partial_t U^n, \partial_t U^n) \to 0$ as $n\to\infty$.

5. Concluding remarks

A nonlinear finite difference scheme is considered to obtain approximate solutions of the viscous Cahn-Hilliard equation. The existence of the numerical solution, and the unconditional stability and error analysis of the scheme are shown. This scheme inherits mass conservation and dissipation property as for the classical solution. We showed that the solution of the viscous Cahn-Hilliard equation decays.

There has been an open question to compare the early stages of evolution of the viscous Cahn-Hilliard equation with the Cahn-Hilliard equation in Novick-Cohen. In order to answer this open question, I will compare the evolution of the viscous Cahn-Hilliard equation with that of the Cahn-Hilliard equation numerically and computationally in a preparing paper.

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