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상수행렬을 이용한 상태 시간지연 시스템의 안정성에 관한 연구

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<요 약>

이 논문에서는 상태방정식에서 상태항에 시간지연이 있는 시스템의 안정성을 다루고 있다. 먼저, 상태 시간지연 시스템의 허축의 근이 어떤 상수 행렬의 고유치가 됨을 보였다. 이 결과를 이용해, 상태지연시간이 구간으로 주어지는 시스템에 대한 안정성의 조건을 유도하였다. 이 안정조건은 상태 시간지연 시스템이 불안해질 때 반드시 허축에 근을 가지는 사실을 이용하였다.

Stability of State Delay Systems Using a Constant Matrix

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<Abstract>

This paper is concerned with stability of state delay systems in which the state delay is known to lie in a certain range. It is shown that if the characteristic equation of a state delay system has pure imaginary roots, then the roots are pure imaginary eigenvalues of a constant matrix. Using this result, a stability condition is proposed through which the stability of state delay systems for all state delays belonging to a known range is guaranteed. The condition is based on the certain fact that the state delay system has pure imaginary roots when the system becomes unstable as the state delay increases.

I. Introduction

There are a number of stability conditions for state delay systems, depending on the nature of the state delay. For example, when the state delay is exactly known, exact stability conditions (based on characteristic equations [1] and Lyapunov equations [2]) can be used. When the state delay is completely unknown, delay independent stability conditions [3-4] can be used. In the real world, however, the state delay is usually not exactly known nor completely unknown, but is only known to lie within a certain range. For this kind of state delay system, stability conditions that guarantee the stability of state delay systems for all state delays belonging to a certain range have been derived in the framework of characteristic polynomials [5] and in the state space framework [6-7].

In this paper, we propose a new necessary and sufficient stability condition that guarantees the stability of state delay systems for all state delays within a certain range. The proposed method is based on computations of pure imaginaryroots of characteristic equations.

The computation of pure imaginary roots is also considered in [8], where the computation of pure imaginary roots is converted to a problem of finding pure imaginary roots of a certain polynomial of an even order. The proposed method is simpler because it requires computing only the eigenvalues of a constant matrix.

II. Computation of pure imaginary roots

Consider the following system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) \tag{1}$$

where $x(t) \in R^n$ is a state. The characteristic equation of (1) is given by

$$det(sI - A_0 - A_1 e^{-sh}) = 0. \tag{2}$$

The stability of (1) is defined as follows [9].

Definition: System (1) is stable if all the roots of its characteristic equation are in the open left half of the complex plane.

Let $\rho(A_0, A_1, h)$ be a real value of the right-most root in (2):

$$\rho(A_0, A_1, h) := \sup\{Re\ s \mid det(sI - A_0 - A_1 e^{-sh}) = 0\} \quad (3)$$

It is known [10] that $\rho(A_0, A_1, h)$ is continuous with respect to changes in the state delay h. Thus, if (1) is stable for $h = h_1$ (i.e., $\rho(A_0, A_1, h) < 0$) and unstable for $h = h_2$ (i.e., $\rho(A_0, A_1, h) \geq 0$), there is $h^* \in (h_1, h_2]$ such that $\rho(A_0, A_1, h^*) = 0$. Hence, the condition $\rho(A_0, A_1, h^*) = 0$ determines the boundary between stability and instability as state delay h varies. The stability condition based on the above argument is stated in the next lemma [5].

Lemma 1: System (1) is stable for $h^* \in [0, h^*)$ if and only if

(i) (1) is stable for $h = 0$.

(ii) For all $w \in [0, \infty)$ and $h \in [0, h^*)$,

$$det(sI - A_0 - A_1 e^{-sh}) \neq 0. \quad (4)$$

A number of methods to check condition (4) have been proposed [5-7]. One of the difficulties in checking (4) is ascribed to the fact that (4) should be checked for all $w \in [0, \infty)$. The next theorem, the main technical contribution of this paper simplifies this difficulty by introducing a constant matrix H whose pure imaginary eigenvalues include all jw satisfying

$$det(sI - A_0 - A_1 e^{-sh}) = 0 \ \text{ for some } h \geq 0.$$

Let E_{ij} denote an $n \times n$ matrix with (i, j)-entry equal to 1 and all other entries equal to zero, and let $E \in R^{n^2 \times n^2}$ be the block matrix $E = [E_{ih}]$ (i.e., the (i, j)-block of E is E_{ij}). Let $D_0 \in R^{n^2 \times n^2}$ and $D_1 \in R^{n^2 \times n^2}$ be defined by

$$\begin{aligned} D_0 &:= Diag(A_0, \cdots, A_0) \\ D_1 &:= Diag(A_1, \cdots, A_1) \end{aligned}$$

respectively. The matrix $H \in R^{n^2 \times n^2}$ is defined as follows:

$$H := \begin{bmatrix} D_0 & D_1 E \\ -D_1 E & -D_0 \end{bmatrix} \quad (5)$$

Theorem 1: $W_1 \subset W_2$, where W_1 is the set defined by

$$W_1 = \{ w \in R \mid w > 0,\ det(jwI - A_0 - A_1 e^{-jwh}) = 0 \text{ for some } h \geq 0 \} \tag{6}$$

and W_2 is the set defined by

$$W_2 = \{ w \in R \mid w > 0,\ det(jwI - H) = 0 \} \tag{7}$$

Proof: We will show that there exists $v \neq 0 \in C^{2n^2}$ such that $(jwI - H)v = 0$ for any $w \in W_1$. From (6), there is $x \neq 0 \in C^n$ such that

$$(jwI - A_0 - A_1 e^{-jwh})x = 0 \tag{8}$$

Let $\alpha \in C^n$ be defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} := x e^{-\frac{jwh}{2}} \tag{9}$$

where $\alpha_i,\ 1 \leq i \leq n$ is a complex number. Let v be defined by

$$v = \begin{bmatrix} u \\ u^* \end{bmatrix} \tag{10}$$

where

$$u = \begin{bmatrix} \alpha_1^* x \\ \alpha_2^* x \\ \vdots \\ \alpha_n^* x \end{bmatrix} \in C^{n^2} \tag{11}$$

We will show that this v satisfies $(jwI - H)v = 0$. From the definition of H , we obtain

$$(jwI - H)v = \begin{bmatrix} (jwI - D_0)u - D_1 E u^* \\ (jwI + D_0)u^* + D_1 E u \end{bmatrix}. \tag{12}$$

Partition $(jwI - H)v$ into $2n$ blocks and let the i-th block of $(jwI - H)v$ be denoted by $r_i \in C^n$. Then r_i, $1 \le i \le n$ is given by

$$
\begin{aligned}
r_i &= (jwI - A_0)a_i^* x - A_1(E_{1i}a_i + E_{2i}a_2 + \cdots + E_{ni}a_n)x^* \\
&= (jwI - A_0)a_i^* a e^{\frac{jwh}{2}} - a_i^* A_1 a e^{-\frac{jwh}{2}} \\
&= a_i^* e^{\frac{jwh}{2}} (jwI - A_0 - A_1 e^{-jwh})a \\
&= a_i^*(jwI - A_0 - A_1 e^{-jwh})x = 0, \quad 1 \le i \le n.
\end{aligned}
$$

The last equality is from (8). Since $r_{i+n} = -r_i^*$, $1 \le i \le n$, (see (12)), we have $r_i = 0$, $n+1 \le i \le 2n$. Hence, $(jwI - H)v = 0$, where $v \ne 0$ (since $x \ne 0$). ■

Due to Theorem 1, the condition (4) is simplified to a check over the finite set W_2 instead of $w \in [0, \infty)$. From the structure of H, if λ is an eigenvalue of H, $-\lambda$ is also an eigenvalue of H. Hence, there are at most n^2 (the number of eigenvalues of H is $2n^2$) possible values of w in W_2.

II. A necessary and sufficient stability condition

In this section, we provide a necessary and sufficient stability condition for (1) based on Theorem 1. To use Theorem 1 in a stability condition, first notice that with stability for $h=0$, we have $det(-A_0 - A_1) \ne 0$, and thus, $det(jwI - A_0 - A_1 e^{-jwh})$ is nonzero for $w=0$, $h \ge 0$. Thus it is only necessary to compute h (if existing) satisfying $det(jwI - A_0 - A_1 e^{-jwh}) = 0$ for some $w > 0$. For this computation, we introduce a function $\gamma_h(A_0, A_1, w)$ defined as:

$$
\gamma(A_0, A_1, w) := \begin{cases} \min G, & G \ne \emptyset \\ \infty, & G = \emptyset \end{cases} \tag{13}
$$

where

$$
G(A_0, A_1, w) = \{h \in R \mid det(jwI - A_0 - A_1 e^{-jwh}) = 0, \; h \ge 0\}.
$$

We can compute γ by using the generalized eigenvalue $\lambda(jwI - A_0, A_1)$ defined by [11]

$$
\lambda(A, B) = \{z \in C \mid det(A - zB) = 0\}.
$$

To have $G \neq \emptyset$, one of the generalized eigenvalues should have a magnitude of 1, since in the definition of $\lambda(A, B)$, we have $z = e^{-jwh}$, where $A = (jwI - A_0)$ and $B = A_1$. From the generalized eigenvalue of magnitude 1, γ_h can be computed by equating e^{-jwh} with the generalized eigenvalue. If $det(jwI - A_0) \neq 0$, the computation is simplified to the eigenvalue computation of a matrix $(jwI - A_0)^{-1}A_1$.

Remark 1: We note that if $w \in W_2$ and $\gamma_h(A_0, A_1, w) < \infty$, then $w \in W_1$. If $\gamma_h(A_0, A_1, w) < \infty$ for all $w \in W_2$, then we have $W_1 = W_2$.

The next theorem is a direct consequence of Lemma 1 and Theorem 1

Theorem 2: System (1) is stable for $h \in [0, h^*)$ and unstable for $h = h^*$ if and only if

(i) (1) is stable for $h = 0$.

(ii) $h^* = \min_{w \in W_2} \gamma_h(A_0, A_1, w)$, where W_2 is defined in (7).

Proof: Theorem 2, together with the definition of γ_h, ensures that (ii) of Theorem 1 is equivalent to (ii) of Lemma 1. ■

Theorem 2 is concerned with the stability of (1) only for $h \in [0, h^*]$. The stability for $h > h^*$ cannot be stated using Theorem 2 except that (1) is unstable for $h = h^* + \frac{2pi}{w}N$, $N = 1, 2, \cdots$, since $e^{-jwh} = e^{-jwh^*}$ for any such h.

IV. Conclusion

In this paper, we have proved that if the characteristic equation of a state delay system has pure imaginary roots, the roots are pure imaginary eigenvalues of a constant matrix. This result is the main technical contribution of this paper. Based on this result, we have proposed a stability condition for state delay systems that guarantees stability for all state delays within a certain range.

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