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Simpler Transformations to a Linear Programming Equivalent and Redundancy of Assurance Region (AR) Conditions in AR-IDEA (Imprecise DEA)

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<Abstract>

While assuming exact data in the ordinary data envelopment analysis (DEA), the development of imprecise data envelopment analysis (IDEA) permits us to deal with imprecise data as well as exact data in DEA. It has been extended to the incorporation of assurance region (AR) and cone-ratio envelopment approaches to DEA, referred to as AR-IDEA. These developments have also shown how to transform the IDEA and AR-IDEA models into ordinary linear programming equivalents via scale transformations and variable alterations plus, in some cases, introducing dummy variables. In the present paper, we show one simpler approach for achieving linear programming equivalents only by variable alterations without rescaling as well as introducing dummy variables. We also provide findings in the use of imprecise data and AR conditions in DEA. This points out that some AR conditions are redundant in effecting the efficiency ratings under AR-IDEA.

AR-IDEA에서 선형계획모델로의 간략한 전환방법 개발 및 AR제약의 퇴화성질 발견

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<요 약>

기존 DEA(Data Envelopment Analysis)에서는 모든 자료(data)가 정확한 값으로 주어져야 하는 반면, IDEA(Imprecise DEA)의 개발은 정확한 값을 가진 자료뿐만 아니라 불완전한 자료 처리를 가능하게 하였다. AR-IDEA는 IDEA의 확장으로써 IDEA에 확신영역(Assurance Region, AR)을 첨가한 것이다. 또한 비선형계획모델인 IDEA 및 AR-IDEA를 동일한 해를 갖는 선형계획모델로 전환하는 방법도 개발되었다. 본 전환방법은 주어진 자료의 척도변환(scale transformations)과 변수변환(variable alterations)을 거쳐야하며, 때로는 가변수(dummy variables)의 도입이 필요함으로 복잡하다고 할 수 있다. 본 논문에서는 이러한 복잡한 과정을 거치지 않고 오직 변수변환만을 통하여 선형계획모델을 얻을 수 있는 간략한 전환방법을 개발한다. 또한 특수한 (그러나 현실에 자주 등장하는) 유형의 확신영역 제약식이 AR-IDEA모델에 개입되었을 경우에 효율성점수(efficiency ratings)에 전혀 영향을 미치지 않는다는 퇴화성질(redundancy)을 발견한다.

1. Introduction

Cooper *et al.* [5] developed IDEA (Imprecise DEA) as a body of concepts and methods which makes it possible to treat imprecise as well as exact data with ordinary DEA. Imprecise data refers to such data that are known only to satisfy ordinal relations or to lie within prescribed bounds. With IDEA, these data are transformed first into nonlinear (and non-convex) problems and then transformed into linear programming equivalents for which suitable algorithms are already available.

Cooper *et al.* [5] also showed how inequality conditions on multiplier *variables* could be treated in this manner. This made it possible to include Assurance Region (AR) conditions of Thompson *et al.* [9, 10] in this same kind of IDEA formulation for treating *data*. This extension also embraces the cone-ratio envelopments developed by Charnes *et al.* [2, 3]. See also Brockett *et al.* [1]. Thus, all of these approaches can be unified in an extension of IDEA which we refer to as AR-IDEA.

Still further extensions and applications were subsequently made. For instance, Cooper *et al.* [6] showed how strict as well as weakly ordered data could be used to effect efficiency evaluations. They then made an application to a Korean mobile telecommunication company in order to provide a concrete example and to illustrate how these methods can be employed. Kim *et al.* [8] applied the developed IDEA and AR-IDEA methods to evaluating efficiencies of telephone offices in Korea.

A limitation noted by R.M. Thrall, as cited in Cooper *et al.* [5], involved the assumption that there exists at least one Decision Making Unit (DMU) which has *only* the maximal value in its input or output data column. Cooper *et al.* [7] referred to such DMU as column maximum DMU (CMD) and confirmed that CMDs need not be present for some imprecise data e.g., the data that lie within upper and lower bounds.

Namely Cooper *et al.* [5] assumed the presence of CMDs for all input and output data columns so that the transformations were made in IDEA and AR-IDEA. Therefore, this limitation was subsequently removed in Cooper *et al.* [7] by introducing a *dummy* variable in a manner that does not otherwise alter the procedures or affect the solutions obtained.

However, we find that the resulting procedures are rather complex. Hence it is one purpose of the present paper to simplify these procedures. Namely we show that it is possible to obtain the desired linear programming equivalents of the nonlinear IDEA and AR-IDEA problems by simple variable alterations in a relatively straightforward manner that extends Cooper *et al.* [5, 6, 7]. Another purpose is to show that some AR conditions are redundant in effecting the efficiency ratings when some special imprecise data are given together in the AR-IDEA model.

The topics to be discussed in this paper are developed in the following order. The next section sets forth some of the models and some of the concepts that underlie AR-IDEA. We then present a simpler method of the transformation only by using variable alterations. This is followed by a redundancy of some AR conditions in AR-IDEA. The final section concludes this paper.

2. AR-IDEA models and methods

As shown in Cooper *et al.* [5], the AR-IDEA model is represented in the following form:

$$\left.\begin{aligned} \max\ \pi_0 &= \sum_{r=1}^{s}\mu_r y_{r0} \\ \text{s.t.}\quad & \sum_{r=1}^{s}\mu_r y_{rj} - \sum_{i=1}^{m}\omega_i x_{ij} \le 0, \quad j=1,\dots,n \\ & \sum_{i=1}^{m}\omega_i x_{i0} = 1 \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \mathbf{y}_r &= (y_{rj}) \in D_r^+, \quad r=1,\dots,s \\ \mathbf{x}_i &= (x_{ij}) \in D_i^-, \quad i=1,\dots,m \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} \mu &= (\mu_r) \in A^+ \\ \omega &= (\omega_i) \in A^- \end{aligned}\right\} \tag{1.3}$$

Here, y_{rj}, x_{ij} respectively represent the amounts of the rth output (r = 1,...,s) and the ith input (i = 1,...,m) for each DMU$_j$ (j = 1,...,n). The y_{r0}, x_{i0} data represent the outputs and inputs for DMU$_0$, the DMU$_j$ to be evaluated.

The sets D_r^+, D_i^- in (1.2) represent constraints of imprecise data for the vector of output variables $\mathbf{y}_r = (y_{r1},...,y_{rn})$ and input variables $\mathbf{x}_i = (x_{i1},...,x_{in})$. The D_r^+ can include constraints, for example, $y_{rj}^- \leq y_{rj} \leq y_{rj}^+$; $y_{rj} - y_{r,j+1} \geq \alpha_{rj}$; $y_{rj} \geq \alpha_{rj} y_{r,j+1}$ where y_{rj}^-, y_{rj}^+ and α_{rj} are positive constants to be specified in advance. We further note that the D_r^+ can include arbitrary linear inequalities that are consistent. The D_i^- follows similarly for inputs.

The sets A^+, A^- in (1.3) represent AR bounds on the multiplier variables μ_r, ω_i. An example is

$$A^+ = \{(\mu_r) \in \Re^s \mid \beta_r^- \leq \mu_r / \mu_1 \leq \beta_r^+, \quad r = 2,...,s\} \tag{2}$$

where β_r^-, β_r^+ represent fixed lower and upper bounds for these output multipliers. The set A applies similarly to input multipliers. Note that a common intention of AR bounds is to ensure positivity of the variables and the AR bounds in the form of (2) also ensure positivity. Thus, throughout this paper, we assume all multipliers are to be positive.

As shown in Cooper *et al.* [7], AR-IDEA model (1) is reduced to an ordinary linear programming problem. This is done by the following three steps:

Step 1. *Scale transformations (or normalizations)* for all the data.
Step 2. *Introduction of dummy variables* to ensure the presence of CMDs.
Step 3. *Variable alterations.*

For brevity, in this paper we do not provide the detailed descriptions on these procedures — see [5, 6, 7].

3. Simpler Transformations

We now show that the desired linear programming equivalent of nonlinear AR-IDEA model (1) can be obtained by simple variable alterations like those in Step 3, without taking into account Steps 1 and 2, above. Before proceeding we assume, without loss of generality, that the sets D_r^+, D_i^- in (1.2) include constraints of arbitrary linear imprecise data which we write

$$\begin{aligned} D_r^+ &= \{\mathbf{y}_r \in \Re^n \mid \mathbf{H}_r^+ \mathbf{y}_r^{\mathrm{T}} \leq \mathbf{h}_r^+\}, \quad r = 1,...,s \\ D_i^- &= \{\mathbf{x}_i \in \Re^n \mid \mathbf{H}_i^- \mathbf{x}_i^{\mathrm{T}} \leq \mathbf{h}_i^-\}, \quad i = 1,...,m. \end{aligned} \tag{3}$$

The $\mathbf{H}_r^+$, $\mathbf{H}_i^-$ are $k_r^+ \times n$, $k_i^- \times n$ matrices where k_r^+, k_i^- signify the number of constraints on the output and input data variables. The $\mathbf{h}_r^+$, $\mathbf{h}_i^-$ are column vectors of which dimensions are k_r^+, k_i^-. So these D_r^+, D_i^- respectively represent the permissible values of output and input data variables satisfying the systems of linear constraints in (3).

For AR-IDEA model (1), we now introduce new variables Y_{rj}, X_{ij} and let

$$\begin{aligned} Y_{rj} &= y_{rj}\mu_r, \quad r = 1,\dots,s;\ j = 1,\dots,n \\ X_{ij} &= x_{ij}\omega_i, \quad i = 1,\dots,m;\ j = 1,\dots,n. \end{aligned} \tag{4}$$

Based on the assumption that all the multipliers are positive, thus the equations in (4) can be changed to

$$\begin{aligned} y_{rj} &= Y_{rj}/\mu_r, \quad r = 1,\dots,s;\ j = 1,\dots,n \\ x_{ij} &= X_{ij}/\omega_i, \quad i = 1,\dots,m;\ j = 1,\dots,n. \end{aligned} \tag{5}$$

This implies that the transformation we desire is completed, which we summarize in the following theorem:

Theorem 1. Let $\mathbf{Y}_r = (Y_{r1},\dots,Y_{rn})$ for each r and $\mathbf{X}_i = (X_{i1},\dots,X_{in})$ for each i. (i) The constraints on the data as in (3) can then be converted into

$$\begin{aligned} B_r^+ &= \{(\mathbf{Y}_r, \mu_r) \in \mathfrak{R}^{n+1} \mid \mathbf{H}_r^+\mathbf{Y}_r^{\mathrm{T}} \le \mu_r \mathbf{h}_r^+\}, \quad r = 1,\dots,s \\ B_i^- &= \{(\mathbf{X}_i, \omega_i) \in \mathfrak{R}^{n+1} \mid \mathbf{H}_i^-\mathbf{X}_i^{\mathrm{T}} \le \omega_i \mathbf{h}_i^-\}, \quad i = 1,\dots,m. \end{aligned} \tag{6}$$

(ii) Model (1) can thus be transformed into the following LP problem:

$$\left.\begin{aligned} &\max\ \pi_0 = \sum_{r=1}^{s} Y_{r0} \\ &\text{s.t.}\quad \sum_{r=1}^{s} Y_{rj} - \sum_{i=1}^{m} X_{ij} \le 0, \quad j = 1,\dots,n \\ &\qquad\ \ \sum_{i=1}^{m} X_{i0} = 1 \end{aligned}\right\} \tag{7.1}$$

$$\left.\begin{aligned} (\mathbf{Y}_r, \mu_r) &\in B_r^+, \quad r = 1,\dots,s \\ (\mathbf{X}_i, \omega_i) &\in B_i^-, \quad i = 1,\dots,m \end{aligned}\right\} \tag{7.2}$$

$$\left.\begin{aligned} \mu &\in A^+ \\ \omega &\in A^- \end{aligned}\right\} \tag{7.3}$$

with all variables to be non-negative. The sets B_r^+, B_i in (7.2) are defined in (6). ■

The proof of Theorem 1 can be done by the equations in (4) and (5). Thus, we simply achieve the linear programming equivalent to (1) without normalizing and introducing dummy variables as was done in Cooper *et al.* [7].

In addition, when we take into account the normalization of the original data (as in Step 1 of Section 2), we need to prove that there exists the column maximum value (as a denominator) in each of the input and output data columns. We also need to prove that the equivalence holds between the original and rescaled models, as in Theorem 1 of Cooper *et al.* [7]. Such proofs are not necessary in the present developments. Due to the normalization, it was also needed to rescale the original AR conditions as in (1.3) for use in the linear programming problems of Cooper *et al.* [7]. In contrast, no change is made for the original AR conditions in the present linear programming problem as shown in (7.3).

3. Redundancy of AR conditions

In this section, we consider model (7) assuming that the multipliers μ_r, ω_i satisfy (2). This can then be represented by

$$\left.\begin{aligned} &\max\ \pi_0 = \sum_{r=1}^{s} Y_{r0} \\ &\text{s.t.}\quad \sum_{r=1}^{s} Y_{rj} - \sum_{i=1}^{m} X_{ij} \le 0, \quad j = 1,\dots,n \\ &\qquad\ \ \sum_{i=1}^{m} X_{i0} = 1 \end{aligned}\right\} \quad (8.1)$$

$$\left.\begin{aligned} &\mathbf{H}_r^+ \mathbf{Y}_r^{\mathrm{T}} \le \mu_r \mathbf{h}_r^+, \quad r = 1,\dots,s \\ &\mathbf{H}_i^- \mathbf{X}_i^{\mathrm{T}} \le \omega_i \mathbf{h}_i^-, \quad i = 1,\dots,m \end{aligned}\right\} \quad (8.2)$$

$$\left.\begin{aligned} &\beta_r^- \le \mu_r / \mu_1 \le \beta_r^+, \quad r = 2,\dots,s \\ &\gamma_i^- \le \omega_i / \omega_1 \le \gamma_i^+, \quad i = 2,\dots,m \end{aligned}\right\} \quad (8.3)$$

Here, examples of the constraints for output data variables in (8.2) are

$$\text{Fixed bounds}: \quad y_{rj}^{-}\mu_r \le Y_{rj} \le y_{rj}^{+}\mu_r \tag{9.1}$$
$$\text{Ratio bounds}: \quad y_{rj}^{-}Y_{r1} \le Y_{rj} \le y_{rj}^{+}Y_{r1};\ Y_{r1} = \mu_r \tag{9.2}$$
$$\text{Strict orders}: \quad Y_{rj} - Y_{r,j+1} \le -\alpha_{rj}\mu_r \tag{9.3}$$
$$\text{Weak orders}: \quad Y_{rj} - Y_{r,j+1} \le 0 \tag{9.4}$$
$$\text{Multiplied orders}: \quad \delta_{rj}Y_{rj} - Y_{r,j+1} \le 0 \tag{9.5}$$
$$\text{Difference ranks}: \quad Y_{rj} - Y_{r,j+1} \le Y_{r,j+1} - Y_{r,j+2}. \tag{9.6}$$

We first consider imprecise data given in (9.4) through (9.6). A common feature of these imprecise data represents $\mathbf{h}_r^+ = \mathbf{0}$, so homogenous linear inequalities $\mathbf{H}_r^+\mathbf{Y}_r^T \le \mathbf{0}$ are then constructed in the form of (8.2). Under this situation we find that some AR bounds are redundant in effecting the AR-IDEA efficiency. This finding is provided in the following Theorem 2, Corollary and Remark:

Theorem 2. (i) Assume that we have $s-1$ AR bounds for the output multiplier variables as in (8.3), i.e., $\beta_r^- \le \mu_r/\mu_1 \le \beta_r^+$, $r = 2,\ldots,s$. If $\mathbf{h}_k^+ = \mathbf{0}$ for any k, $2 \le k \le s$ and $\mathbf{h}_r^+ \ne \mathbf{0}$, $r \ne k$, then $\beta_k^- \le \mu_k/\mu_1 \le \beta_k^+$ among the given $s-1$ AR bounds is redundant in effecting the solution to (8). (ii) Assume that we have $\gamma_i^- \le \omega_i/\omega_1 \le \gamma_i^+$, $i = 2,\ldots,m$ for the input multiplier variables as in (8.3). If $\mathbf{h}_l^- = \mathbf{0}$ for any l, $2 \le l \le m$ and $\mathbf{h}_i^- \ne \mathbf{0}$, $i \ne l$, then $\gamma_l^- \le \omega_l/\omega_1 \le \gamma_l^+$ is redundant in effecting the solution to (8).

Proof. Per (i), if $\mathbf{h}_k^+ = \mathbf{0}$ for any k, then the variable μ_k is vanished in (8.2). Thus in (8.1) and (8.2) we can set $\mu_k = \tau$, where τ is an arbitrary positive number. Then we can change $\beta_k^- \le \mu_k/\mu_1 \le \beta_k^+$ to $\beta_k^-\mu_1 \le \tau \le \beta_k^+\mu_1$ while the other AR bounds remain unchanged. Since τ is also vanished in (8.1) and (8.2), $\beta_k^-\mu_1 \le \tau \le \beta_k^+\mu_1$ is redundant. Thus, $\beta_k^- \le \mu_k/\mu_1 \le \beta_k^+$ is redundant in effecting the optimal objective value of (8).

To complete the proof, we now consider other AR bounds not to be revealed among the $s-1$ AR bounds but implied by $\beta_k^- \le \mu_k/\mu_1 \le \beta_k^+$. These are the upper-lower bounds for the variables μ_k/μ_2, μ_k/μ_3, ..., μ_k/μ_s. In similar manner, it can be shown that these AR bounds are also redundant. Therefore the AR bound $\beta_k^- \le \mu_k/\mu_1 \le \beta_k^+$ is not necessary in (8).

A similar proof can be done for part (ii). ■

Corollary. In (8), (i) if $\mathbf{h}_k^+ = \mathbf{0}$ for all k, $2 \le k \le s$, then all the AR bounds for output multipliers are redundant. (ii) If $\mathbf{h}_l^- = \mathbf{0}$ for all l, $2 \le l \le m$, then all the AR bounds for the input multipliers are redundant. ■

Remark. Note that, for output multipliers, the $s-1$ AR bounds in the form of (8.3) is sufficient to represent the marginal rates of substitutions about all the pairs of outputs. Because, as mentioned in the proof of Theorem 2, these imply other AR bounds that are not revealed among the given s - 1 AR bounds. However, sometimes AR bounds can be obtained in a different form from those in (8.3) in which $\mu 1$ is the common denominator (or referent). For example, $1\leq \mu_2/\mu_1\leq 2$; $3\leq \mu_4/\mu_2\leq 5$; $2 \leq \mu_3/\mu_4\leq 4$. However these can be expressed in the form of (8.3) so that $1\leq \mu_2/ \mu_1\leq 2$; $6\leq \mu_3/\mu_1\leq 40$; $3\leq \mu_4/\mu_1\leq 10$. Thus Theorem 2 can now be applied in such cases. If $\mathbf{h}_4^+ = \mathbf{0}$ and the other $\mathbf{h}_r^+ \neq \mathbf{0}$, then only the last AR bound $3\leq \mu_4/\mu_1\leq 10$ is not necessary.

When assessing the data of *qualitative* input or output factors for use with AR-IDEA, their absolute measurement units (such as number, dollar, time and so on) are generally absent. The thus obtained data can be imprecise and comparative (or relative) across DMUs, for example, as weak orders or multiplied orders as in (9.4) or (9.5). Note that difference rankings as in (9.6) may also be elicited after obtaining (9.4). Such data are then represented in homogenous linear inequalities, *viz.*, $\mathbf{H}_r+\mathbf{Y}_r^T \leq \mathbf{0}$, $\mathbf{H}_i^-\mathbf{X}_i^T\leq \mathbf{0}$ (due to $\mathbf{h}_r^+ = \mathbf{0}$, $\mathbf{h}_i^- = \mathbf{0}$).

For such qualitative factors without their absolute measurement units, it may relatively be difficult to assess AR bounds for their multipliers which can be regarded as prices or costs per the unit amount of outputs or inputs. As an evidence we refer to an application of AR-IDEA to a Korean mobile telecommunication company presented in Cooper *et al.* [6], where three inputs and three outputs are used. For one of three inputs, "Level of management for facilities and customers" which is qualitative, these data are given in the form of weak orders as in (9.4). It was also not easy to assess AR bounds associated with this input multiplier in the application of Cooper *et al.* [6]. So they have dropped this multiplier and used AR bounds associated with the other multiplier variables in the form of (8.3), which are sufficient to effect the same (or desired) AR-IDEA efficiency because of Theorem 2 we developed. Therefore, this finding will be helpful to the potential users of AR-IDEA.

4. Conclusions

We have provided a simpler approach to the transformation of AR-IDEA into linear programming equivalents. This was done only by the variable alteration. We also showed a redundancy of some AR conditions when a special form of imprecise data represented by homogenous inequalities is given together.

On the other hand, as noted in Chen *et al.* [4], there are other ways besides the variable alteration developed in this paper. It is possible to do this by first choosing exact data from imprecise data given in advance. If all the exact data of inputs and outputs under consideration can be chosen from the given imprecise data, then this makes possible to use already available DEA computer codes in the thus generated (ordinary) DEA model.

In spite of such computational convenience, it does not imply that we can always accomplish ordinary DEA via Chen et al. approach to IDEA and AR-IDEA problems. This approach works generally for some special types of imprecise data such as (i) fixed bounds as shown above, (ii) ratio bounds that are known in the form of (9.2), and (iii) weak orders in the form of (9.4). It is also important to note that all the components in the respective set of constraints D_r^+, D_i^- should be given by the same kind of imprecise data among (i), (ii) and (iii). When the components of D_r^+ or D_i^- are consist of mixtures of them, it may be difficult (or nearly impossible, as far as we know) to accomplish a generalized method for choosing exact data from D_r^+ or D_i^-.

Moreover, we note other troubles underlying Chen et al. approach. This is associated with choosing exact data from weak orders as in (iii). Consider that the determined exact data are used in AR-IDEA model (10). Apparently these weak ordinal data belong to homogenous linear inequalities which we have denoted by $\mathbf{h}_r^+ = \mathbf{0}$ or $\mathbf{h}_i^- = \mathbf{0}$. However these are no longer homogenous inequalities or equalities since these are already reduced to exact data and then at least one data variable has a *positive* value. Thus, *generally* it is not true that the given AR bound associated with the multiplier for this weak ordinal data can be redundant in effecting AR-IDEA efficiency in (10). However, following Theorem 2 and Corollary we developed in the present paper, it is possible that the corresponding AR bound is redundant.

In contrast, our approach can deal with *arbitrary* linear imprecise data including (9.1) through (9.6). Without choosing exact data, imprecise data given in advance are incorporated into DEA models as constraints, the thus IDEA and AR-IDEA are created, and these are transformed into linear programming equivalents by methods we have developed. Moreover no trouble is involved in the issue of redundancy of AR bounds as happened in Chen *et al.* [4] approach.

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