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On the Cesàro Sequence Spaces

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〈**Abstract**〉

The Cesàro sequence spaces Ces(p), $1<p<\infty$ and Ces(∞) was introduced by J.S. Shiue (1969) and he showed that they have an analytic structure like spaces l_p and l_∞.

In this paper we consider various topological relationships between the Cesàro sequence spaces Ces(p), $1<p<\infty$ and we give conditions of a nature which are equivalent to the statement that Ces(p)=Ces(q).

Cesàro Sequence Spaces에 관한 연구

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〈요 약〉

Cesàro sequence spaces Ces(p)와 Ces(∞)는 J.S. Shiue (1969)가 처음 소개한 sequence space로서 l_p space와 l_∞ space들과 유사한 해석적 구조를 가지고 있음이 연구되었다.

본 논문에서는 Cesàro sequence spaces Ces(p), $1<p<\infty$들 사이의 몇가지 위상적 관계를 규명하고 Ces(p)=Ces(q)가 성립할 필요충분 조건을 구하였다.

I. Introduction.

Let S be the linear space of all infinite sequences $x=\{x_n\}$ of complex numbers over complex field. The Cesàro sequence spaces Ces(p), $1<p<\infty$, and Ces(∞) are defined in [2] rerpectively as

$$\text{Ces}(p)=\{\{x_n\}:\{x_n\}\in S, \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p<\infty\}$$

with a norm defined by

$$\|x\|_{c(p)}=\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}},$$

and

$$\text{Ces}(\infty)=\{\{x_n\}:\{x_n\}\in S, \sup_n\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|\right\}<\infty\}$$

with a norm defined by

$$\|x\|_{c(\infty)}=\sup_n\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|\right\}.$$

We write $u(p)$ for the topology on Ces(p) induced by the norm $\|\cdot\|_{c(p)}$, $v(\infty)$ for the topology on Ces(∞) induced by the $\|\cdot\|_{c(\infty)}$ (see [2]), and we also write $w(p)$ for the topology on l_p space induced by the norm $\|\cdot\|_p$, $1<p<\infty$.

It has been shown ([2], [5]) that the Cesàro sequence spaces (Ces(p), $\|\cdot\|_{c(p)}$) and (Ces(∞), $\|\cdot\|_{c(\infty)}$) are Banach spaces and l_p spaces are contained in Ces(p) with $1<p<\infty$.

In this paper we investigate some relationships between the Ces(p) spaces and relationships between l_p and Ces(p), $1<p<\infty$.

We now employ some notation which will be used in this paper. The set C is the set of all complex numbers. If convenience, we identify e^n with the vector

$$e^n=(0,\cdots,0,1,0,\cdots),$$

where the nonzero entry is in n-th position. If

A is a countable dense subset of C, then the set Φ_A is the set of all sequence in A with at most finitely many nonzero entries. For $x=\{x_n\}\in S$, the sequence $\{\sigma(x)_m\}$ is the sequence of the averaging means $(\sigma(x))_m=\frac{1}{m}\sum_{i=1}^{m}|x_k|$ of the absolute value sequence of $\{x_n\}$.

II. The Cesàro sequence spaces Ces(p), $1<p<\infty$.

In this section we consider various relationships between the Ces(p) spaces, $1<p<\infty$. In Theorems 2.4 and 2.6. we give conditions of a topological nature which are equivalent to the statement that $\mathrm{Ces}(p)=\mathrm{Ces}(q)$.

Definition 2.1. A sequence $\{\alpha_n\}$ in a Banach space E is called a Schauder basis if each $x\in E$ has a unique representation $x=\sum_{n=1}^{\infty}x_n\alpha_n$, where the series converges in E, and the coefficient functionals are continuous.

Lemma 2.2. If $1<p<\infty$, $\{e^n\}$ forms a Schauder basis for Banach space $(\mathrm{Ces}(p), \|\cdot\|_{c(p)})$. Consequently, $(\mathrm{Ces}(p), \|\cdot\|_{c(p)})$ is separable.

Proof. We first show that if $x=\{x_n\}\in\mathrm{Ces}(p)$, then $\sum_{k=1}^{n}x_ke^k$ converges to x in Ces(p). To show this, let $\tau(x)_m$ be the averaging mean for the absolute value sequence of $x-\sum_{k=1}^{n}x_ke^k=(0,\cdots,0,x_{n+1},x_{n+2},\cdots)$. Then

$(\tau(x))_m=0$ for $1\leqslant m\leqslant n$.

$$(\tau(x))_{n+k}=\frac{|x_{n+1}|+\cdots+|x_{n+k}|}{n+k}\quad\text{for } k=1,2,\cdots.$$

Clearly, $(\tau(x))_{n+k}\leqslant(\sigma(x))_{n+k}$. Hence the norm of $x-\sum_{k=1}^{n}x_ke^k$ in Ces(p) is

$$\left(\sum_{k=1}^{\infty}(\tau(x))^p_{n+k}\right)^{\frac{1}{p}}\leqslant\left(\sum_{k=1}^{\infty}(\sigma(x))^p_{n+k}\right)^{\frac{1}{p}}$$
$$=\left(\sum_{m=n+1}^{\infty}(\sigma(x))^p_m\right)^{\frac{1}{p}}.$$

Since the tails of a convergent series approach 0, and since $\sum_{m=1}^{\infty}(\sigma(x))^p_m$ is convergent, we have

$$\lim_{n\to\infty}\left\|x-\sum_{k=1}^{n}x_ke^k\right\|_{c(p)}=0.$$

Hence $x=\sum_{n=1}^{\infty}x_ne^n$ and the representation is unique. The continuity of the coefficient functionals follows from an inequality

$$\|x\|_{c(p)}\geqslant|x_n|\left(\sum_{k=0}^{\infty}\left(\frac{1}{n+k}\right)^p\right)^{\frac{1}{p}}\text{ for each } n=1,2,\cdots.$$

Lemma 2.3. If $1<p\leqslant q<\infty$, then the following statements are true.

(1) $\mathrm{Ces}(p)\subseteq\mathrm{Ces}(q)$,

(2) the inclusion map $(\mathrm{Ces}(p),u(p))\longrightarrow(\mathrm{Ces}(q),u(q))$ is continuous,

(3) the inclusion map $(l_p, w(p))\longrightarrow(\mathrm{Ces}(p),u(p))$ is continuous,

(4) Ces(p) is dense in $(\mathrm{Ces}(q),u(q))$,

(5) l_p is dense in $(\mathrm{Ces}(p),u(p))$.

Proof. (1) : If $x=\{x_n\}\in\mathrm{Ces}(p)$, then $\{(\sigma(x))_n\}\in l_p$. Since

$$\left(\sum_{n=1}^{\infty}(\sigma(x))^q_n\right)^{\frac{1}{q}}\leqslant\left(\sum_{n=1}^{\infty}(\sigma(x))^p_n\right)^{\frac{1}{p}}\text{ (see [1], p.4)},$$

it follows that $\{(\sigma(x))_n\}\in l_q$. This means that $x=\{x_n\}\in\mathrm{Ces}(q)$.

(2) : For given $\varepsilon>0$, if $\|x\|_{c(p)}<\varepsilon$, then $\|x\|_{c(q)}\leqslant\varepsilon$. This means that the inclusion map $(\mathrm{Ces}(p),u(p))\longrightarrow(\mathrm{Ces}(q),u(q))$ is continuous at 0. Since the inclusion map is linear, it is continuous everywhere [6, p.37].

(3) : The proof follows from the inequality [1, p.248]

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}\leqslant\frac{p}{p-1}\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}.$$

(4) : Let A be a countable dense subset of C. Then it can be shown that Φ_A is dense in $(\mathrm{Ces}(q),u(q))$ and $\Phi_A\subset\mathrm{Ces}(p)\subset\mathrm{Ces}(q)$. Hence the set Ces($p$) is dense in $(\mathrm{Ces}(q),u(q))$.

(5) : The proof follows from the fact that $\Phi_A\subset l_p\subset\mathrm{Ces}(p)$, where A is the same set as in the proof of (4).

Theorem 2.4. If $1<p\leqslant q<\infty$, then the following four statements are equivalent:

(1) $u(p)$ is the topology induced on Ces(p) by $u(q)$.

(2) If $\{x^N\}(N=1,2,\cdots)\in\mathrm{Ces}(p)$ and $x^N\longrightarrow0$ in $u(q)$ then $x^N\longrightarrow0$ in $u(p)$.

(3) Ces(p) is closed in $(\mathrm{Ces}(q),u(q))$.

(4) $\mathrm{Ces}(p)=\mathrm{Ces}(q)$.

Proof. (1)$\Longrightarrow$(3) : If (1) is true, then $\|\cdot\|_{c(p)}$

and $\|\cdot\|_{c(q)}$ give the same definition of Cauchy sequence in Ces(p). So, by Lemma 2.2, (Ces(p), $\|\cdot\|_{c(q)}$) is complete. Hence (3) follows.

(3)$\Longrightarrow$(4) : The proof follows immediately from Lemma 2.3(4).

(4)$\Longrightarrow$(1) : The proof follows from Lemma 2.3(2), the fact that the inclusion map Ces(p) $\longrightarrow$Ces(q) is now onto and the open mapping theorem [4, p. 195].

(1)$\Longrightarrow$(2) : The proof is clear from the fact two norms $\|\cdot\|_{c(p)}$, $\|\cdot\|_{c(q)}$ induce the same topology on Ces(p).

(2)$\Longrightarrow$(1) : If (2) is true, then the inclusion map (Ces(p), $u(q)$)$\longrightarrow$(Ces(p), $u(p)$) is continuous at 0, hence continuous everywhere. Hence we have $u(p)\subset u(q)\cap$Ces(p). By Lemma 2.3(2), the inclusion map (Ces(p), $u(p)$)$\longrightarrow$(Ces(q), $u(q)$) is continuous. Hence $u(q)\cap$Ces(p)$\subset u(p)$. Therefore, (1) follows.

Remark 2.5. The spaces Ces(p) for distinct p are distinct. For the proof of this, see [2, p. 153].

Theorem 2.6. If $1<p, q<\infty$, then (1), (2) and (3) are equivalent, and (4), (5) and (6) are equivalent:

(1) Ces(p)$\subseteq$Ces(q).

(2) $u(p)$ induces a topology on Ces(p)$\cap$Ces(q) as fine as $u(q)$ does.

(3) If $\{x^N\}(N=1,2,\cdots)\in$Ces(p)$\cap$Ces(q) and $x^N\longrightarrow 0$ in $u(p)$, then $x^N\longrightarrow 0$ in $u(q)$.

(4) Ces(p)=Ces(q).

(5) $u(p)$ and $u(q)$ induce the same topology on Ces(p)$\cap$Ces(q).

(6) If $\{x^N\}(N=1,2,\cdots)\in$Ces(p)$\cap$Ces(q) then $x^N\longrightarrow 0$ in $u(p)$ if and only if $x^N\longrightarrow 0$ in $u(q)$.

Proof. Clearly it suffices to show the equivalence of (1), (2) and (3).

(1)$\Longrightarrow$(2) : If (1) is true, then Ces(p)=Ces(p)$\cap$Ces(q). Hence by Lemma 2.3(2) the topology $u(p)$ on Ces(p) is as fine as the topology $u(q)$ on Ces(p), *i.e.* $u(q)\cap$Ces(p)$\subset u(p)$.

(2)$\Longrightarrow$(3) : The proof is obvious.

(3)$\Longrightarrow$(1) : By Lemma 2.3(1), it is enough to show that $p<q$. Suppose $q<p$; then by Lemma 2.3(1) we have Ces(q)$\subset$Ces(p). Thus Ces(p)$\cap$Ces(q)=Ces(q). Hence it follows from Theorem 2.4 (equivalence (2) and (4)) that we must have Ces(p)=Ces(q). This implies that $p=q$ (by Remark 2.5), which contradicts the assumption $q<p$. Therefore (1) follows.

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