울산대학교 연구논문집 제18권(자연과학·공학편) 제 1 호 pp. 47~49. 1987.
UOU Report Vol. 18. (Natural Science and Engineering) No. 1. pp. 47~49. 1986,
University of Ulsan.

On the Stablity of AF C^*-algebra

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(Received Septermber 30, 1986)

〈Abstract〉

Let A be an AF C^*-algebra. By the trace on A and the ordered group $K_0(A)$ we show that an AF C^*-algebra A is stable if it has no non zero finite trace.

AF C^*-대수의 안정성

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(1986. 9. 30 접수)

〈국문초록〉

A가 AF C^*-대수일 때 trace와 K_0군을 이용하여 유한 trace가 없는 AF C^*-대수가 안정함을 보임.

In this note we study the stablity of the AF C^*-algebra A by means of the trace on A and the ordered group $K_0(A)$. A C^*-algebra A is an AF C^*-algebra if it is an inductive limit of the increasing sequence of the finite dimensional C^*-algebras. The study of AF C^*-algebra has been begun by Glimm[4] and Diximier[2], and much expanded by Bratteli. [1] Elliot showed that the ordered group $K_0(A)$ with a certain hereditary generating subset of the positive cone is complete isomorphism invariant for AF C^*-algebra A. Hence $K_0(A)$ is the very useful tool to characterize AF C^*-algebras. [3].

Definition 1. Let A be a C^*-algebra, and let e, f be projections in A. The projections e and f are *-equivalent, written $e \stackrel{*}{\sim} f$ if there is an element $u \in A$ such that

$$u = euf, \quad uu^* = e \quad u^*u = f$$

Then $\stackrel{*}{\sim}$ is an equivalence relation on the set of projections in A. Define $P(A) = \bigcup_{n=1} \{$Projections in $\mathrm{Mn}(A)\}$ where $\mathrm{Mn}(A)$ is the set of the $n \times n$ matrices over A.

Definition 2. The projections e, f in $P(A)$ are stably *-equivalent, written $e \stackrel{*}{\approx} f$ if $e \oplus g \stackrel{*}{\sim} f \oplus g$ for some g in $P(A)$.

To define $e \stackrel{*}{\sim} f$ in $P(A)$ is to mean that $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \stackrel{*}{\sim} \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ for some suitable sized zero matrices. Then $\stackrel{*}{\approx}$ is an equivalence relation on $P(A)$ and the abelian group $(P(A)/\stackrel{*}{\approx}, \oplus)$ is denoted by $K_0(A)$. For any C^*-algebra A we set

$$K_0(A)^+ = \{[e] \mid e \in P(A)\}.$$

Every element of $K_0(A)$ has the forms $[e]-[f]$ for e, f in $P(A)$. For any x, y in $K_0(A)$, we define $x \leq y$ if and only if $y - x \in K_0(A)^+$. Then if A is an AF C^*-algebra, $\leq$ is the partial order on $K_0(A)$. Let $\phi : A \to B$ be a *-homomorphism. Then ϕ induces a map $\mathrm{Mn}(A) \to \mathrm{Mn}(B)$ such that $(a_{ij})_{n \times n} \to (\phi(a_{ij}))_{n \times n}$, also denoted by ϕ. Then ϕ induces a map $P(A)$ into $P(B)$, furthermore induces a group homomorphism

$K_0(\phi): K_0(A) \to K_0(B)$ such that $K_0(\phi)([e]-[f]) = [\phi(e)] - [\phi(f)]$.

Theorem 1. [*Elliot*]. *Let A and B be AF C^*-algebras. Then the following conditions are equivalent*;

i) $A \simeq B$ as C^*-algebras

ii) $(K_0(A), [1_A]) \simeq (K_0(B), [1_B])$ as partially ordered abelian groups with order unit.

So the above theorem showes that AF C^*-algebra can be classified up to isomorphism by K_0, considered as a partially ordered abelian group with order unit. We call an element x in A positive if x is normal and $Sp(x) \subset R_+$. Denote A_+ the set of all positive elements in A. Then the set A_+ is a closed real cone in A_{sa}, the set of self adjoint elements in A. A_{sa} becomes partially ordered real vector space by defining $x \leq y$ whenever $y - x \in A_+$

Definition 3. Let B be a C^*-subalgebra of A. B is called hereditary if $0 \leq x \leq y$ and $y \in B$ imply $x \in B$ for each $x \in A$.

Now we define the trace τ on a C^*-algebra A and the set of traces has been long recognized the very useful invariant of the algebra.

Definition 4. Let A be a C^*-algebra. A trace on A is a function $\tau: A_+ \to [0, \infty]$ such that

i) $\tau(\alpha x) = \alpha\tau(x)$ if $x \in A_+$, $\alpha \in R_+$

ii) $\tau(x+y) = \tau(x) + \tau(y)$ if $x, y \in A_+$

iii) $\tau(u^*xu) = \tau(x)$ for all unitary u in $\tilde{A}$.

Here $\tilde{A}$ means the C^*-algebra with unit contaning A as a closed ideal. Since the trace τ has the value ∞, naturally we have come to consider the set

$$A_+^\tau = \{x \in A_+ : \tau(x) < \infty\}.$$

We call the trace τ a finite trace if the linear span of A_+^τ is A_+. And the trace τ is semifinite if the linear span of A_+^τ is an essential ideal in A. Hence we can normalize the finite trace τ with $\tau(1_A) = 1$ for a C^*-algebra A. Let $T(A)$ be the set of normalized finite traces on A.

Lemmal 1. *Let B be a hereditray C^*-subalgebra of A. Then every finit trace τ on B has a extension to a lower semi-continuous, semi-finite trace on A.*

Proof. See in [2].

In [5] another equivalence relation, applicable to all positive operators, was defined in a C^*-algebra A.

Definition 5. Let A be a C^*-algebra and x, y be in the positive cone A_+. x is equivalent to y, written $x \sim y$, if, there is a sequence (u_n) in A with

$$x = \Sigma u_n^* u_n, \quad y = \Sigma u_n u_n^*$$

where the sums are norm convergent.

Then $\sim$ is a countably additive equivalence relation. We write $x \propto y$ if $x \sim z \leq y$ for some z in A_+.

Definitcon 6. A positive element x in A is finite if for each y in A_+

$$y \leq x \text{ and } y \sim x \text{ imply } x = y.$$

We say that A is a finite C^*-algebra if every element in A_+ is finite.

Lemma 2. *Let A be an AF C^*-algebra, p and q be projections of A. If $\tau(p) < \tau(q)$ for all non-zero traces on A, then $[p] < [q]$ in $K_0(A)$.*

Proof. Let (A_n) be an increasing sequence of finite dimensianal C^*-algebras and A be the norm closure of $\bigcup_{n=1}^{\infty} A_n$ We may assume that p and q lie in a finite dimensional subalgebra A_1 of A. If $\tau(p) < \tau(q)$ for all traces on A, then for sufficiently large n $\tau(p) < \tau(q)$ for all traces on A_n. If not; let e be the identity of A_1. We can choose a finite trace τ_n on A_n such that $\tau_n(e) = 1$ and $\tau_n(p) \geq \tau_n(q)$ for each n. Then τ_n is a normalized trace on eA_ne. Put $B = eAe$. Since B has a unit e, the state space of B is weak*-compact. Let φ_n be a state extension of τ_n to A and φ be a any weak* limit point of the sequence$(\varphi_n|_B)$. Since $\bigcup_1^{\infty} eA_ne$ is dense in B, φ is a tracial state on B. Since B is hereditary, by Lemma 1 we can extend φ to a trace τ on A. Futhermore $\tau(p) \geq \tau(q)$ and this is contradiction to the hypothesis. Since An is finite algebra, $[p] \leq [q]$ in $K_0(A_n)$ for some n. Hence $[p] < [q]$ in $K_0(A)$.

Let K be the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.

Definition 7. Let A, B be C^*-algebras. A is stably isomorphic to B if $A\otimes K \simeq B\otimes K$.

Then $A\otimes K$ is clearly stable, because $(A\otimes K)\otimes K \simeq A\otimes(K\otimes K)\simeq A\otimes K$.

Theorem 2. *Let A be an AF C^*-algebra with no nonzero finite trace. Then A is stable.*

Proof. Choose a finite projection e of A, and let $B=eAe$. Then B is an AF C^*-algebra, futhermore finite algebra. We want to show that A is isomorphic to the stable algebra $B\otimes K$. By Theorem 1, it suffices to show that the hereditary generating subset X of $K_0(B)_+$ corresponding to A is all of $K_0(B)_+$. Let $\{e_n\}$ be an approximate identity of A consisting of projections with $e_1=e$. By Lemma 1, we can define a real-valued continuous function ϕ_n on $T(B)$ by $\phi_n(\tau)=\tau(e_n)$ for all $\tau\in T(B)$. Since A has no finite trace, $\sup_n \phi_n(\tau)=\infty$ for any $\tau\in T(B)$. Let q be a projection in B and φ be the function in $T(B)$ with $\varphi(\tau)=\tau(q)$ for all τ in $T(B)$. Since $C(T(B))$ is compact and $\sup_n \phi_n(\rho)=\infty$ for all $\tau\in T(B)$, there exist n such that $\phi_n(\tau)>\sup \varphi(\tau)$ for all τ. Hence $\tau(e_n)>\tau(q)$ for all τ in $T(B)$. Hence by Lemema 2 $[q]<[e_n]$, therefore $[q]\in X$.

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