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On Weakly θ-continuous Functions

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〈**Abstract**〉

The concept of θ-closed sets is due to N. V. Velicko [12]. Complement of θ-closed set is said to be θ-open set. Mamta Deb [2] has studied weakly continuous functions [6] and seen that if $f: X \longrightarrow Y$ is a function for which the inverse image of every θ-open set is open, then f is not necessary weakly continuous, however, weakly continuous function implies this type of function.

In this note we named these type of functions by weakly θ-continuous functions and present its study.

Weakly θ-연속함수에 관하여

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〈요　　약〉

논문 [12]와 [2]에서, 이 논문에서 명명한 Weakly θ-연속함수이면 Weakly 연속함수가 될 필요가 없으나 그 역과정은 성립됨이 알려졌다.

이 논문에서는 이 Weakly θ-연속함수에 관한 연구이다.

I. Preliminary

For a topological space X and $A \subset X$, the θ-closure [12] of A denoted as $cl_\theta\ A$, is $\{x \in X$: every closed neighbourhood of x meets $A\}$. The subset A is θ-closed [12] if $cl_\theta\ A = A$. Similary, a subset A of a space X is said to be θ-open [2] if for every point $x \in$ A there exists an open set G such that $x \in G \subset clG \subset A$ (cl denotes the closure operator). A subset is regular open if it is the interior of the closure of itself. Complement of a regular open set is called regular closed. The concept of weakly continuity is due to N. Levine [6]. A function $f: X \longrightarrow Y$ is said to be weakly continuous if for each $x \in X$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset cl\ V$. Fomin [4] defined θ-continuous functions on replacing $f(U)$ by $f(cl\ U)$ in the definition of weakly continuous functions.

It is known that θ-continuity implies weakly continuity but converse is not true. Weakly θ-continuous functions generalize weakly continuous functions. In the present note we define weakly θ-continuous functions and present their study.

II. Properties of weakly θ-continuous functions

Definition 1.1: *A function* $f: X \longrightarrow Y$ *is said*

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to be weakly θ-continuous if and only if inverse image of each θ-open set in Y is open in X.

Theorem 1.1. Let $f: X \longrightarrow Y$ be a function from a topological space X into a topological space Y. Then the following statements are equivalent;

(a) f is weakly θ-continuous.

(b) For each $x \in X$ and each θ-open set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.

(c) If V is a θ-closed subset of Y, then $f^{-1}(V)$ is closed subset of X.

Proof. (a)$\Rightarrow$(b) Let $x \in X$ and V be a θ-open set of Y containing $f(x)$. Then $f^{-1}(V)$ is an open set containing x and $f(f^{-1}(V)) \subset V$.

(b)$\Rightarrow$(a) If V is a θ-open subset of Y, then for each $x \in f^{-1}(V)$, V is a neighbourhood of $f(x)$. Hence there is an open neighbourhood U of x such that $f(U) \subset V$. Thus $f^{-1}(V)$, being a neighbourhood of each of its points, is open.

(a)$\Rightarrow$(c) Let $K \subset Y$ be a θ-closed set. Then $Y-K$ is a θ-open set and therefore, $f^{-1}(Y-K)=X-f^{-1}(K)$ is open. So $f^{-1}(K)$ is closed in X.

(c)$\Rightarrow$(a) Let $V \subset Y$ be a θ-open set. Then $Y-V$ is a θ-closed set and therefore, $f^{-1}(Y-V)=X-f^{-1}(V)$ is closed. So $f^{-1}(V)$ is open in X.

Lemma 1.1. A space X is Hausdorff if and only if $\{x\}$ is θ-closed for all $x \in X$.

Theorem 1.2. Let $f: X \longrightarrow Y$ be a weakly θ-continuous, one to one function. If Y is Hausdorff, then X is T_1.

Proof. Let $x \in X$. Then by Lemma 1.1, $\{f(x)\}$ is θ-closed in Y. Hence by Theorem 1.1 (c), $f^{-1}(f(x))=\{x\}$ is closed. Therefore X is T_1.

Theorem 1.3. Let $f: X \longrightarrow Y$ be any funtion, $\wedge$ and $\wedge'$ be index sets. Then the following statements are true;

(a) If f is weakly θ-continuous and $A \subset X$. Then $f/A: A \longrightarrow Y$ is weakly θ-continuous.

(b) If $\{U_\alpha : \alpha \in \wedge\}$ is an open cover of X and if for each α, $f_\alpha = f/U_\alpha$ is weakly θ-continuous, then f is weakly θ-continuous.

(c) If $\{F_\beta : \beta \in \wedge'\}$ is a locally finite closed cover of X and if for each β, $f_\beta = f/F_\beta$ is weakly θ-continuous, then f is weakly θ-continuous.

Proof. (a) Let U be a θ-open subset of Y. Then $f^{-1}(U)$ is open and hence $(f/A)^{-1}(U) = f^{-1}(U) \cap A$ is an open subset of A.

(b) Let U be a θ-open subset of Y. Then $f^{-1}(U) = \cup\{f_\alpha^{-1}(U) : \alpha \in \wedge\}$ and since each f_α is weakly θ-continuous, each $f_\alpha^{-1}(U)$ is open in X and so $f^{-1}(U)$ is open in X.

(c) Let F be a θ-closed subset of Y. Then $f^{-1}(F) = \cup\{f_\beta^{-1}(F) : \beta \in \wedge'\}$. Since each f_β is weakly θ-continuous, by theorem 1.1(c), each $f_\beta^{-1}(F)$ is closed in F_β and hence in X. Again, since $\{F_\beta : \beta \in \wedge'\}$ is locally finite closed cover of X, the collection $\{f_\beta^{-1}(F) : \beta \in \wedge'\}$ is a locally finite collection of closed sets. Thus $f^{-1}(F)$ being the union of a locally finite collection of closed sets is closed.

Definition 1.2. Completely separable, H-closed spaces are called θ-bicompacta. A space is completely separable if every two distinct points have nonintersecting closed neighbourhoods.

Lemma 1.2. Let Y be an open subspace of a space X. If $U \subset Y$ is regular open in Y, then $U = Y \cap \text{Int}\ cl\ U$. (Int denotes the interior operator).

Lemma 1.3. In a θ-bicompacta each regular closed set is θ-closed.

Remark 1.1. A weakly θ-continuous function $f: X \longrightarrow Y$ need not have the restriction $f/A: A \longrightarrow f(A)$ to be weakly θ-continuous. For,

Example 1.1. Let $X=\{a,b,c\}=Y$ and $\mathscr{T}=\{\emptyset, X\}$ and $\mathbf{Q}=\{\emptyset, Y, \{a\}, \{b\}, \{a,b\}\}$. Let $f: (X, \mathscr{T}) \longrightarrow (Y, \mathbf{Q})$ be identity function and $A=\{a,b\}$. Then f is weakly θ-continuous but $f/A: A \longrightarrow f(A)$ is not weakly θ-continuous.

Theorem 1.4. Let f be a weakly θ-continuous function from a space X into a θ-bicompacta Y and $A\subset X$ be such that $f(A)$ is open in Y. Then $f/A: A\longrightarrow f(A)$ is weakly θ-continuous.

Proof. Let $x\in A$ and U be θ-open in $f(A)$ containing $f(x)$. By definition of θ-open sets, there exists an open set V in $f(A)$ such that $f(x)\in V\subset cl_{f(A)}V\subset U$. Let $Int_{f(A)}\ cl_{f(A)}V=W$. Then W is regular open in $f(A)$ and by Lemma 1.2, $W=f(A)\cap Int\ cl\ W$. By Lemma 1.3, $Y-Int\ cl\ W$ is θ-closed, therefore $Int\ cl\ W$ is θ-open in Y. Now,

$$\begin{aligned} f/A^{-1}(W) &= f/A^{-1}(f(A)\cap Int\ cl\ W) \\ &= f/A^{-1}(f(A))\cap f/A^{-1}(Int\ cl\ W) \\ &= A\cap f/A^{-1}(Int\ cl\ W) \\ &= A\cap f^{-1}(Int\ cl\ W). \end{aligned}$$

Since f is weakly θ-continuous, $f^{-1}(Int\ cl\ W)$ is open in X and hence $f/A^{-1}(W)$ is open in the subspace A and $x\in f/A^{-1}(W)$. And $f/A(f/A^{-1}(W))\subset W\subset U$. Hence by Theorem 1.1(b), $f/A: A\longrightarrow f(A)$ is weakly θ-continuous.

Theorem 1.5. If $f: X\longrightarrow Y$ is continuous and $g: Y\longrightarrow Z$ is weakly θ-continuous, then $gof: X\longrightarrow Z$ is weakly θ-continuous.

Proof. Let U be a θ-open subset of Z. Then $g^{-1}(U)$ is open in Y and since f is continuous, $(gof)^{-1}(U)=f^{-1}(g^{-1}(U))$ is open in X.

Theorem 1.6. If $f: X\longrightarrow Y$ is weakly θ-continuous and $g: Y\longrightarrow Z$ is θ-continuous, then $gof: X\longrightarrow Z$ is weakly θ-continuous.

Proof. Let U be a θ-open subset of Z. If $x\in g^{-1}(U)$ then $g(x)\in U$. By definition of θ-open sets, there exists an open set V in Z such that $g(x)\subset V\subset cl\ V\subset U$. By θ-continuity of g there exists an open set W in Y such that $x\in W$ and $g(cl\ W)\subset cl\ V$. Therefore $x\in W\subset cl\ W\subset g^{-1}(U)$. Hence $g^{-1}(U)$ is θ-open in Y and since f is weakly θ-continuous, $(gof)^{-1}(U)=f^{-1}(g^{-1}(U))$ is open in X.

Definition 1.3. Let $f: X\longrightarrow Y$ be any function. Then the function $g: X\longrightarrow X\times Y$, defined by $g(x)=(x,f(x))$ is called the graph function with respect to f.

Lemma 1.4. If $f: X\longrightarrow Y$ is a continuous function, then inverse image of θ-open set is θ-open.

Proof. Let V be θ-open subset of Y. Then for each $x\in f^{-1}(V)$, $f(x)\in V$ and since V is θ-open there exists an open set U such that $f(x)\in U\subset cl\ U\subset V$. $f^{-1}(U)$ is open and $cl\ f^{-1}(U)\subset f^{-1}(cl\ U)\subset f^{-1}(V)$. Therefore $x\in f^{-1}(U)\subset cl\ f^{-1}(U)\subset f^{-1}(V)$. Hence $f^{-1}(V)$ is θ-open in X.

Theorem 1.7. If $f: X\longrightarrow Y$ is a function such that the graph function g is weakly θ-continuous, then f is weakly θ-continuous.

Proof. Let $x\in X$ and V be a θ-open set containing $f(x)$. Since projection map p_Y is continuous, by Lemma 1.4, $p_Y^{-1}(V)$ is θ-open in $X\times Y$. Therefore, by Theorem 1.1(b), there exists an open set U containing x such that $g(U)\subset p_Y^{-1}(V)$. It follows that $p_Y(g(U))=f(U)\subset V$, so that f is weakly θ-continuous.

Lemma 1.5. A compact subset of a Hausdorff space is θ-closed.

Theorem 1.8. Let $f: X\longrightarrow Y$ be a weakly θ-continuous function from a compact space X in a Hausdorff space Y. Then inverse image of compact set is compact.

Proof. Let K be a compact subset of Y. Then by Lemma 1.5, K is θ-closed in Y. By Theorem 1.1(c), $f^{-1}(K)$ is closed in X and since X is compact, $f^{-1}(K)$ is compact.

Theorem 1.9. Let $f: X\longrightarrow Y$ be a weakly θ-continuous onto function. If X is connected so is Y.

Proof. Suppose Y is disconnected, then there exists a proper closed open set K in Y. Then K is both θ-open and θ-closed. And by Theorem 1.1, $f^{-1}(K)$ is open and closed. Also $f^{-1}(K)$ is proper subset of X. This gives a contradiction.

III. Comparision

Definition 2.1. A subset A of a space X is an H-set if every cover of A by sets open in X has a finite subfamily whose closures in X

cover A. H-sets are equivalent to H-closed [9] sets.

Lemma 2.1. *A* function $f: X \longrightarrow Y$ is c-continuous if and only if for each closed and compact subset V of Y, $f^{-1}(V)$ is closed in X.

Lemma 2.2. *A* function $f: X \longrightarrow Y$ is H-continuous if and only if for each closed and H-closed subset V of Y, $f^{-1}(V)$ is closed in X.

Lemma 2.3. *A* function $f: X \longrightarrow Y$ is almost continuous if and only if inverse image of regular open subset of Y is open in X.

Definition 2.2. *A* function $f: X \longrightarrow Y$ is said to be θ-compact if $f^{-1}(K)$ is H-set in X whenever K is H-set in Y.

Remark 2.1. Continuity$\Longrightarrow$almost continuity $\Longrightarrow\theta$-continuity$\Longrightarrow$weakly continuity.

We know that weakly continuity implies weakly θ-continuity, however, converse is not true. It follows from Remark 2.1 that continuity and almost continuity imply weakly θ-continuity, however, the converse in both cases may not be true.

Also, the function defined in Example 1.1 is weakly θ-continuous but neither c-continuous nor H-continuous.

Thus by Examples 2.2 and 1.1 weakly θ-continuous function is independent of c-continuous function and of H-continuous functions. And by Examples 2.1 and 2.2 weakly θ-continuous function is independent of θ-compact function.

Example 2.1. Let $f: (R, \mathscr{D}) \longrightarrow (R, \mathscr{T})$ be the identity function with $\mathscr{T}$, the cofinite topology and $\mathscr{D}$, the discrete topology. Then f is weakly θ-continuous but not θ-compact.

Example 2.2. Let $f: (R, \mathscr{T}) \longrightarrow (R, \mathscr{D})$ be the identity function with $\mathscr{T}$, the cofinite topology and $\mathscr{D}$, the discrete topology. Then f is c-continuous, H-continuous and θ-compact but not weakly θ-continuous.

Defintion 2.3. A space in which every closed subset is an H-set is called C-compact.

Lemma 2.4. Each H-set in a completely separable space is θ-closed.

Proof. Let P be an H-set in a completely separable space X. Let $x \in (X-P)$. For each point $y \in P$ there exists a closed neighbourhood W_y not intersecting with some closed neighbourhood W_x of the point x. There exist finite number of sets $W_{y_i}: i=1, 2, \cdots n$ which cover P. The neighbourhood $W_x = \bigcap_{i=1}^{n} Wx_i$ of the point x is closed and does not intersect P. Therefore $X-P$ is θ-open. Hence P is θ-closed.

Theorem 2.1. Let $f: X \longrightarrow Y$ be a weakly θ-continuous function. If Y is completely separable and C-compact, then f is continuous.

Proof. Let F be closed subset of Y, then by definition of C-compactness, F is an H-set in Y. By Lemma 2.4. F is θ-closed. And by Theorem 1.1(c), $f^{-1}(F)$ is closed in X.

Theorem 2.2. Let $f: X \longrightarrow Y$ be a weakly θ-continuous function. If Y is θ-bicompacta then f is almost continuous.

Proof. Let U be regular open subset of Y, then $Y-U$ is regular closed and by Lemma 1.3, U is θ-open. Hence $f^{-1}(U)$ is open. Therefore, by Lemma 2.3, f is almost continuous.

Definition 2.4. *A* space X is said to be almost regular if for every regular closed set F and a point $x \notin F$, there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.

Theorem 2.3. Let $f: X \longrightarrow Y$ be a weakly θ-continuous function. If Y is almost regular then f is almost continuous.

Proof. Let U be a regular open subset of Y and $x \in U$. Since Y is almost regular there exist disjoint open sets M and N containing x and $Y-U$ respectively. And so $Y-N$ is closed and $(Y-N) \subset U$. Therefore $x \in M \subset \mathrm{cl}\, M \subset (Y-N) \subset U$. Hence U is θ-open and so $f^{-1}(U)$ is open. By Lemma 2.3, f is almost continuous.

Theorem 2.4. Let $f: X \longrightarrow Y$ be weakly θ-continuous function. If Y is Hausdorff, then f is c-continuous.

Proof. Let F be closed and compact subset of Y. By Lemma 1.5, F is θ-closed. And so by Theorem 1.1(c), $f^{-1}(F)$ is closed. Therefore by Lemma 2.1, f is c-continuous.

Theorem 2.5. Let $f: X \to Y$ be weakly θ-continuous function. If Y is completely separable then f is H-continuous.

Proof. Let F be closed and H-closed subset of Y. Since H-sets are equivalent to H-closed subsets, by Lemma 2.4, F is θ-closed. And by Theorem 1.1(c), $f^{-1}(F)$ is closed. Hence by Lemma 2.2, f is H-continuous.

Lemma 2.5. A θ-closed subset of an H-closed space is an H-set.

Theorem 2.6. Let $f: X \to Y$ be an H-continuous function. If Y is H-closed, then f is weakly θ-continuous.

Proof. Let F be θ-closed subset of Y. By Lemma 2.5, F is H-closed subset of Y. And by Lemma 2.2, f^{-1} (F) is closed in X.

Theorem 2.7. Let f be a θ-compact function from a Hausdorff space X into an H-closed space Y. Then f is weakly θ-continuous.

Proof. Let F be a θ-closed subset of Y. Then by Lemma 2.5, F is an H-set in Y. By definition of θ-compact function, $f^{-1}(F)$ is an H-set in X. Since H-sets [9] in a Hausdorff space are closed, therefore, $f^{-1}(F)$ is closed. And so by Theorem 1.1(c), f is weakly θ-continuous.

Theorem 2.8. Let f be a weakly θ-continuous function from a C-compact space into a completely separable space Y. Then f is θ-compact.

Proof. Let F be an H-set in Y. Then by Lemma 2.4, f is θ-closed. By Theorem 1.1(c), $f^{-1}(F)$ is closed. And by definition of C-compactness, $f^{-1}(F)$ is an H-set in X. Hence f is θ-compact.

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