UIT Report Vol. 15. No. 2. pp. 367~371. 1984 연구논문집 제15권 제 2 호 pp. 367~371. 1984

Feebly open sets and feebly-continuity in topological spaces

Do Won Lee and Gyu Ihn Chae
Dept. of Mathematics
(Receive April 30, 1984)

〈**Abstract**〉

Maheshwari [6] defined a feebly open set and introduced a notion of feebly-continuity. The purpose of this note is to investigate their properties and to find the necessary and sufficient conditions for a mapping to be feebly continuous.

Feebly 개집합과 feebly 연속성에 대하여

이 도 원 · 채 규 인
수 학 과
(1984. 4. 30 접수)

〈**요 약**〉

[6]에서 Maheshwari가 정의한 feebly개집합에 관한 여러가지 성질과 또 약화된 연속함수, feebly-연속함수들에 관한 여러가지 성질을 알아내고 feebly 연속되기 위한 필요충분 조건을 알아본다.

I. Introduction

S. N. Maheshwari [6] defined a feebly-open set in a topological space. A set A is said to be semiopen [3] if there exists an open set O such that $O \subset A \subset \mathrm{cl}\, O$, where cl denotes the closure operator in a topological space X. A set is semiclosed if its complement is semiopen. The smallest semiclosed set containing A is called the semiclosure of A and denoted by scl A.

A set A is termed feebly open [6] if there exists an open set O such that $O \subset A \subset \mathrm{scl}\, O$. Every open set is feebly open and every feebly open set is semiopen. However, the converses need not be true. A set is feebly closed if its complement is feebly open.

A mapping of a topological space X into a topological space Y is termed feebly continuous if the inverse image of every open set of Y is feebly open in X.

The purpose of this note is to investigate the properties of feebly open sets and feebly continuous mappings.

II. Preliminary

In this section we shall investigate some properties of feebly open sets.

Definition 2.1. A set N in a space X is said to be a feeble neighborhood of a point p in X if there exists a feebly open set A such that $p \in A \subset N$.

Theorem 2.2. Any union of feebly open sets is feebly open.

Theorem 2.3. A set is feebly open iff it is a feeble neighborhood of each of its points.

From Theorem 2.2. it follows that any intersection of feebly closed sets is feebly closed.

Definition 2.4. A point p in X is said to be a feebly interior point of A if A is a feeble neighborhood of p.

The set of all feebly interior points of A is called feeble interior of A and denote it by fint A.

Remark: fint $A \subset A$. The equality may not hold, in general.

Theorem 2.5. A set A is feebly open iff fint $A = A$. Thus fint (fint A) = fint A.

Definition 2.6. A point p in X is said to be a feebly limit point of A if for any feebly open set U containing p, $U \cap (A - p) \neq \emptyset$. The set of all feebly limit points of A is called feebly derived set of A and denoted by fd A.

Remark: Every feebly limit point of A is a limit point of A, but the converse may be false.

Theorem 2.7. A is feebly closed iff it contains its feebly derived set.

Definition 2.8. $A \cup fd$A is defined to be the feeble closure of A and denote it by fcl A.

Theorem 2.9. A set A is feebly closed iff fcl $A = A$. Thus fcl (fcl A) = fcl A.

Theorm 2.10. The feeble closure of A is the smallest feebly closed set containing A.

Theorem 2.11. fint $A = A - fd(X - A)$.

Proof: Let $x \in A - fd(X - A)$. Then $x \notin fd(X - A)$ iff there exists a feebly open set U such that $U \cap (X - A)) = \emptyset$ iff $x \in U \cap A$ iff $x \in$ fint A.

Corollary: (a) $X -$ fint $A =$ fcl$(X - A)$and (b) $X -$ fcl $A =$ fint$(X - A)$.

Theorem 2.12. fcl $A \cup$ fclB $\subset$ fcl$(A \cup B)$ and fcl $(A \cap B) \subset$ fcl $A \cap$ fcl B.

The equality may not, in general, hold.

As a consequence of the above theorems 2.11 and 2.12, it follows that if $A \subset B$, then fint $A \subset$ fint B and fcl $A \subset$ fcl B.

We shall omit the proofs since they can be proved easily.

Theorem 2.13. If $A \subset X$, then int $A \subset$ fint $A \subset$ sint $A \subset A \subset$ scl $A \subset$ fcl $A \subset$ cl A, where sint denotes semi-interior operator in X.

Proof: The results are easy from definitions and we refer to [5].

Ⅲ. Feebly open sets

By F.O.(X) we shall denote the class of all feebly open sets in a space X.

Theorem 3.1. If A is feebly open in a space X and $A \subset B \subset$ scl A, then B is feebly open.

Proof: Since A is feebly open, there exists an open set O such that $O \subset A \subset$ scl$O$. Then $O \subset B$ and scl $A \subset$ scl(scl O) = scl$O$. Thus $B \subset$ scl$O$ and hence, B is feebly open.

Theorem 3.2. Let $\mathscr{A} = \{A_\alpha\}$ be a collection of sets in a space (X, T) such that (1) $T \subset \mathscr{A}$ and (2) $A \in \mathscr{A}$ and $A \subset B \subset$ scl A, then $B \in \mathscr{A}$. Then F.O.$(X) \subset \mathscr{A}$. Thus F.O.(X) is the smallest class of sets satisfying (1) and (2).

Proof: Let V be in F.O.(X). Then there exists an open set O such that $O \subset V \subset$ scl O. Then $O \in \mathscr{A}$ by (1) and so, $V \in \mathscr{A}$ by (2).

Intersection of two semiopen sets need not be semiopen and union of two semiclosed sets may not be semiclosed. However, referring to [2], we can see that if A is semiopen in X and U is open in X, then $A \cap U$ is semiopen in X and if A is semiopen in X and U is open in X, then $A \cap U$ is semiopen in U. And so, if B is semiclosed and F is closed, then $B \cup F$ is semiclosed in X (or in$F$).

Theorem 3.3. If U and O are open in X, and $O \subset A \subset$ scl O (A is feebly open), then $(U \cap O) \neq \emptyset$ implies that $(U \cap A) \neq \emptyset$.

Proof: $O \subset (X - U)$ implies that scl $O \subset (X - U)$ since U is semiopen. But $A \subset$ scl O, thus $O \subset (X - A)$ implies that $A \subset (X - U)$. The proof is complete.

Lemma: Let A be a subset of a space X and U be open in X. Then $U \cap \text{scl } A \subset \text{scl } (U \cap A)$.

Proof: For $x \in U \cap \text{scl } A$ and any semiopen neighborhood $V(x)$ of x, $U \cap V(x)$ is again a semiopen neighborhood of x and hence $(U \cap A) \cap V(x) \neq \emptyset$, so that $x \in \text{scl } (U \cap A)$.

Theorem 3.4. If U is open and A is feebly open, then $(U \cap A)$ is feebly open.

Proof: Since A is feebly open, there is an open set O with $O \subset A \subset \text{scl } O$. Then $U \cap O \subset U \cap A \subset U \cap \text{scl } O$. From the above Lemma, $U \cap \text{scl } O \subset \text{scl } (U \cap O)$. Hence $U \cap A$ is feebly open.

Remark: If $A \subset Y \subset X$ and if Y is semiopen in X, then $\text{scl}_X A \cap Y \subset \text{scl}_Y A$ where scl_X denotes the semiclosure operator in X. The equality holds if Y is open.

Theorem 3.5. Let $A \subset Y \subset X$ where Y is a subspace of X and let $A \in F.O.(X)$. If Y is semiopen in X, then $A \in F.O.(Y)$.

Proof: Let $A \in F.O.(X)$. Then for some openset O, $O \subset A \subset \text{scl}_X O$. Now $O \subset Y$ and $O = O \cap Y \subset A \cap Y \subset \text{scl}_X O \cap Y$. Thus $O \subset A \subset \text{scl}_Y O$ by Remark. Since $O = O \cap Y$ is open in Y, $A \in F.O.(Y)$.

Remark: The converse of Theorem 3.5 is false, as shown by

Example 3.6. Let R be the usual space of reals, $Y = \{0\}$, and $A = \{0\}$. Then A is feebly open in Y but A is not feebly open in R.

Theorem 3.7. Let Y be an open subspace of a space X. Then every feebly open set in Y is feebly open in X.

Proof: Let A be feebly open in Y and Y be open in X. Then for some open set O_Y of Y, $O_Y \subset A \subset \text{scl}_Y O_Y$. Let O be an open set of X such that $O_Y = O \cap Y$. Then $O \cap Y \subset A \subset \text{scl}_Y (O \cap Y) = \text{scl}_X (O \cap Y) \cap Y \subset \text{scl}_X (O \cap Y)$. Hence A is feebly open in X by Theorem 3.1.

Lemma: Let O be open in X. Then $\text{scl} O - O$ is nowhere dense in X.

Proof: The result is obvious since $\text{cl} O - O$ is nowhere dense.

Theorem 3.8. Let (X, T) be a space and $A \in F.O.(X)$. Then $A = O \cup B$ where (1) $O \in T$, (2) $O \cap B = \emptyset$ and (3) B is nowhere dense.

Proof: Since $A \in F.O.(X)$, $O \subset A \subset \text{scl } O$ for some open set O of X. But $A = O \cup (A - O)$. Let $B = A - O$. Then $B \subset \text{scl} O - O$ and thus is nowhere dense by Lemma. Hence $A = O \cup B$, and (1) and (2) follow.

Remark: The converse of Theorem 3.8 is false as shown by

Example 3.9. Let R be the space of reals and $A = \{x \mid 0 < x < 1\} \cup \{2\}$. Then $A \notin F.O.(X)$ although (1), (2) and (3) in Theorem 3.8. hold.

Remark: $(\text{scl } A) \times (\text{scl } B) = \text{scl } (A \times B)$.

Theorem 3.10. Let X_1 and X_2 be spaces and $X_1 \times X_2$ be the topological product. Let $A_1 \in F.O.(X_1)$ and $A_2 \in F.O.(X_2)$. Then $A_1 \times A_2$ is feebly-open in $X_1 \times X_2$.

Proof: For each open set O_i of X_i $(i = 1, 2)$, by Theorem 3.8, $A_i = O_i \cup B_i$ where $B_i \subset \text{scl}_{X_i} O_i - O_i$. Then $A_1 \times A_2 = (O_1 \times O_2) \cup (B_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times B_2)$. But $(O_1 \times O_2)$ is open in $X_1 \times X_2$ and $(B_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times B_2) \subset (\text{scl}_{X_1} O_1) \times (\text{scl}_{X_2} O_2) = \text{scl}_{X_1 \times X_2} (O_1 \times O_2)$. Hence $A_1 \times A_2 \in F.O.(X_1 \times X_2)$.

IV. Feebly-continuous mappings

A mapping $f : X \longrightarrow Y$ from a space X into a space Y is said to be semi-continuous if for each open set V of Y, $f^{-1}(V)$ is semiopen in X.

Definition 4.1. A mapping $f : X \longrightarrow Y$ is termed feebly-continuous if for every open set V of Y, $f^{-1}(V)$ is feebly-open in X

Remark: Continuity $\Longrightarrow$ Feebly-continuity $\Longrightarrow$ Semi-continuity. But the implications may not be conversible, as shown by

Example 4.2. Let R be the space of reals. Let a mapping $f : R \longrightarrow R$ be defined by $f(x) = 1$ if $x \leqq 0$ and $f(x) = 0$ if $x > 0$. Then f is semicontinus, but neither continuous nor feebly continuous.

Example 4.3. Let $X=\{a, b, c\}$ be the space with a topology $T=\{X, \emptyset, \{a\}\}$. Define $f: X \longrightarrow X$ by $f(a)=f(c)=a$ and $f(b)=b$. Then f is feebly-continuous, but not continuous.

Theorem 4.4. Let X and Y be spaces and $f: X \longrightarrow Y$ be a mapping. Then the followings are equivalent.

(a) f is feebly continuous

(b) The inverse image of each closed set of Y is feebly closed.

(c) For each $p \in X$ and each neighborhood $V(f(p))$ of $f(p)$ in Y, there exists a feeble neighborhood $U(p)$ of p in X such that $f(U(p)) \subset V(f(p))$.

Proof: Since for any $A \subset Y$, $f^{-1}(Y-A)=X-f^{-1}(A)$, one has $(a) \Longleftrightarrow (b)$.

$(a) \Longrightarrow (c)$: Since $f^{-1}(V(f(p)))$ is feebly-open, we can use it for $U(p)$.

$(c) \Longrightarrow (a)$: Let V be an open set of Y containing $f(p)$ in X. Then by (c), there exists a feebly open set $U(p)$ containing p in X such that $f(U(p)) \subset V$. And so, $p \in U(p) \subset f^{-1}(V)$. Therefore, $f^{-1}(V)$ is a union of feebly open sets in X. Hence by Theorem 2.2, $f^{-1}(V)$ is feebly-open in X. Thus (a) holds.

Theorem 4.5. Let a mapping $f: X \longrightarrow Y$ be feebly-continuous and Y be a second countable space. Let P be the set of points of discontinuity of f. Then P is of first category.

Proof: Let $p \in P$ and $\{V_n\}$ be a countable open basis for Y. Then, since f is discontinuous at p, there exists a $V_i \in \{V_n\}$ such that $p \in U$ open in X implies $f(U) \not\subset V_i$. By Theorem 4.4-(c), there exists an $A_i \in F.O.(X)$ such that $p \in A_i$ and $f(A_i) \subset V_i$. And by Theorem 3.8, $A_i = O_i \cup B_i$ where O_i is open and $B_i \subset \text{scl}\, O_i - O_i$. Hence $p \notin O_i$ and so $p \in B_i$. Then $P \subset \bigcup_{p \in P} B_i$ and each B_i is nowhere dense. Thus the proof is complete since $\bigcup_{p \in P} B_i$ is of first category.

Remark: The converse of Theorem 4.5 is, in general, false, as shown by

Example 4.6. Let $X=(0,1]$ and $Y=[0,1]$. Let $f(x)=0$ if x is irrational and $f(x)=\frac{1}{q}$ if $x=\frac{p}{q}$ where $(p, q)=1$. Then f is continuous at each irrational, but not continuous at each rational. Thus f is continuous at the irrationals and discontinuous at the rationals. Hence the set of points of discontinuity of f is of first category. But $f: X \longrightarrow Y$ is not feebly continuous because $f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$ is a subset of rationals and so is not feebly open.

Theorem 4.7. Let $f_i: X_i \longrightarrow Y_i$ be feebly continuous for $i=1,2$. Let $f: X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ be defined by $f(x_1, x_2)=(f_1(x_1), f_2(x_2))$. Then f is feebly continuous.

Proof: Let $V_1 \times V_2 \subset Y_1 \times Y_2$ where V_i is open in Y_i for $i=1,2$. Then $f^{-1}(V_1 \times V_2)=f_1^{-1}(V_1) \times f_2^{-1}(V_2)$. But $f_1^{-1}(V_1)$ and $f_2^{-1}(V_2)$ are feebly-open since each f_i is feebly continuous. Thus by Theorem 3.10, $f_1^{-1}(V_1) \times f_2^{-1}(V_2)$ is feebly-open in $X_1 \times X_2$. If V is an open set in $Y_1 \times Y_2$, then $f^{-1}(V)=f^{-1}(\bigcup_\alpha V_\alpha)$ where V_α is of the form $V_{\alpha_1} \times V_{\alpha_2}$. Then $f^{-1}(V)=\bigcup_\alpha f^{-1}(V_\alpha)$ which is feebly-open since $f^{-1}(V_\alpha)$ is feebly-open.

Theorem 4.8. Let $f: X \longrightarrow X_1 \times X_2$ be feebly continuous. Let $f_i: X \longrightarrow X_i$ be defined by $f_i(x)=x_i$ for $x \in X$, $f(x)=(x_1, x_2)$. Then $f_i: X \longrightarrow X_i$ is feebly-continuous for $i=1,2$.

Proof: It is enough to show that $f_1: X \longrightarrow X_1$ is feebly continuous. Let V_1 be open in X_1. Then $V_1 \times X_2$ is open in $X_1 \times X_2$ and $f^{-1}(V_1 \times X_2)$ is feebly open in X. And $f_1^{-1}(V_1)=f^{-1}(V_1 \times X_2)$ and thus $f_1: X \longrightarrow X_1$ is feebly continuous.

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