울산대학교 연구논문집 제18권(자연과학·공학편) 제 1 호 pp. 43~46. 1987.
UOU Report Vol. 18. (Natural Science and Engineering) No. 1. pp. 43~46. 1987,
University of Ulsan.

Note on Quaternionic CR-Submanifolds of a Quaternionic Projective Space

Shin, Yong-Ho · Kang, Tae-Ho
Dept. of Mathematics
(Received September 25, 1986)

〈Abstract〉

Let M be an n-dimensional CR-submanifold of a quaternionic space form. Then we shall investigate the relations of parallel f-three-structure tensors on M induced from the quaternionic structure and characterize the submanifold M.

四元數射影空間의 四元數 CR-部分多樣體에 대한 小考

申 龍 浩·姜 泰 浩
數 學 科
(1986. 9. 25 접수)

〈要　約〉

4원수 사영공간의 CR-부분다양체상에 유도된 구조텐서가 평행일때 이들 사이의 성질 및 부분공간의 형태를 연구하였다.

Ⅰ. Introduction

The quaternionic projective space, its non-compact dual and the quaternion number space Q^m are three important examples of quaternionic Kaehlerian manifold. Let M be a quaternionic CR-submanifold in a quaternionic Kaehlerian manifold $\bar{M}$ (see Ⅱ). Then M is called a QR-product if locally M is the Riemannian product of an invariant submanifold and a totally real submanifold of $\bar{M}$[1].

In this paper, we shall find some partial answers in order for such a submanifold to be a QR-product.

Ⅱ. CR-submanifolds of quaternionic Kaehlerian manifolds

Let $\bar{M}^{4m}$ be a 4m-dimensional quaternionic Kaehlerian manifold and its quaternionic Kaehlerian structure be denoted by $(\langle,\rangle, V)$. Then there is a canonical local basis $\{I, J, K\}$ of 3-dimensional vector bundle V consisting of tensors of type (1,1) over $\bar{M}^{4m}$ such that

(2.1) $$I^2=J^2=K^2=-Id,\ IJ=-JI=K,\ JK=-KJ=I,\ KI=-IK=J,$$

where Id is the identity tensor field of type (1,1) on $\bar{M}^{4m}$. Moreover the local tensor fields I, J, K are almost Hermitian with respect to $\langle , \rangle$ and the equations

$$(2.2)\qquad \begin{aligned} \overline{\nabla}_{\overline{X}} I &= r(\overline{X})J - q(\overline{X})K \\ \overline{\nabla}_{\overline{X}} J &= -r(\overline{X})I + p(\overline{X})K \\ \overline{\nabla}_{\overline{X}} H &= q(\overline{X})I - p(\overline{X})J \end{aligned}$$

are valid for any vector field $\overline{X}$ on $\overline{M}^{4m}$, $\overline{\nabla}$ being the Riemannian connection determined by $\langle , \rangle$, where p, q and r are local 1-forms[2]. As is well known, every quaternionic projective space QP^m of real dimension $4m$ is a quaternionic space form with constant Q-sectional curvature 4 by a suitable normalization.

Let M be an n-dimensional submanifold isometrically immersed in a $4m$-dimensional quaternionic Kaehlerian manifold $\overline{M}^{4m}$.

We denote by the same symbol $\langle , \rangle$ the Riemannian metric tensor field induced from that of $\overline{M}^{4m}$.

For any vector field X tangent to M, we put

$$(2.3)\qquad \phi_r X = P_r X + F_r X, \quad (r=1,2,3).$$

where $\phi_1 = I$, $\phi_2 = J$, $\phi_3 = K$ and $P_r X$ are the tangential parts and $F_r X$ the normal parts of $\phi_r X$ respectively. Then P_r is an endomorphism on the tangent bundle TM and F_r is a normal bundle valued 1-form on TM. Similarly, for any vector field η normal to M, we put

$$(2.4)\qquad \phi_r \xi = t_r \xi + f_r \xi, \quad (r=1,2,3),$$

where $t_r\xi$ are the tangential parts and $f_r\xi$ the normal parts of $\phi_r\xi$ respectively.

For any vector field Y tangent to M, we have from (2.3)

$$(2.5)\qquad \langle P_r X, Y\rangle = -\langle P_r Y, X\rangle, (r=1,2,3).$$

Similarly, for any vector field normal to M, we have from (2.4)

$$(2.6)\qquad \langle f_r\xi, \eta\rangle = -\langle f_r\eta, \xi\rangle.$$

We have also from (2.3) and (2.4)

$$(2.7)\qquad \langle F_r X, \xi\rangle + \langle t_1\xi, X\rangle = 0.$$

A distribution $D: x \mapsto D_x \subseteq T_x M$ is called an invariant distribution if we have $\phi_r(D) \subseteq D$, $r=1,2,3$. A submanifold M in a quaternionic Kaehlerian manifold $\overline{M}$ is called a quaternionic CR-submanifold if it admits a differentiable invariant distribution D such that its orthogonal complementary distribution $D^\perp$ is totally real, i.e., $\phi_r(D_x^\perp) \subseteq T_x^\perp M$, $r=1,2,3$. for any x in M, where $T_x^\perp M$ denotes the normal space of M in $\overline{M}$ at x.

A submanifold M in a quaternionic Kaehlerian manifold $\overline{M}$ is called an invariant subamanifold(respectively, a totally real submanifold if dim $D_x^\perp = 0$ (respectively, dim $D_x = 0$). A quaternionic CR-submanifold is said to be proper if it is neither totally real nor invariant.

A submanifold M of a quaternionic Kaehlerian manifold $\overline{M}^{4m}$ to be a quaternionic CR-submanifold, it is necessary and sefficient that $P_1 = P_2P_3 = -P_3P_2$, $P_2 = P_3P_1 = -P_1P_3$, $P_3 = P_1P_2 = -P_2P_1$ and $F_rP_r = 0$, $(r=1,2,3)$ hold on M Moreover, $\{P_1, P_2, P_3\}$ is an f-three-structure in M and $\{f_1, f_2, f_3\}$ is an f-three-structure in the normal bundle (see[4]).

Ⅲ. Some results

Let $\imath: M \to \overline{M}$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+p)$-dimensional Riemannian manifold $\overline{M}$. Denote by $\langle , \rangle$ the inner product induced from the Riemannian metric of $\overline{M}$. Let ∇ and $\overline{\nabla}$ be the connections on M and $\overline{M}$, respectively. The second fundamental form H of the immersion i is given by

$$(3.1)\qquad H(X,Y) = \overline{\nabla}_X Y - \nabla_X Y,$$

where X and Y are vector fields tangent to M. For a vector field ξ normal to M and X tangent to M, we put

$$(3.2)\qquad \overline{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where $-A_\xi X$ and $\nabla_X^\perp\xi$ denote the tangential and normal components of $\overline{\nabla}_X\xi$, respectively. Then, we have

$$(3.3)\qquad \langle H(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

For the second fundamental form H, the vector $H(X, X)$ is called a normal curvature vector in the direction of a unit vector X. If every normal curvature vector has the same length for any unit vector X at $x \in M$, the immersion is said to be isotropic at x. If the immersion i is isotropic at any point on M, then the immersion i is said to be isotropic[3].

Now, differentiating covariantly (2.3) and (2.4) and using (3.1), (3.2), (2.3) and (2.4), we obtain

Lemma 1. *Let M be a submanifold in a quaternionic Kaehlerian manifold $\bar{M}$. Then the following equations are valid.*

$$(3.4)\quad \begin{cases} (\nabla_Y P_1)X = A_{F_1X}Y + r(Y)P_2X \\ \qquad -q(Y)P_3X + t_1H(X,Y), \\ (\nabla_Y P_2)X = A_{F_2X}Y + p(Y)P_3X \\ \qquad -r(Y)P_1X + t_2H(X,Y), \\ (\nabla_Y P_3)X = A_{F_3X}Y + q(Y)P_1X \\ \qquad -p(Y)P_2X + t_3H(X,Y), \\ ('\nabla_Y F_1)X = r(Y)F_2X - q(Y)F_3X \\ \qquad +f_1H(X, Y) - H(P_1X, Y), \\ ('\nabla_Y F_2)X = p(Y)F_3X - r(Y)F_1X \\ \qquad +f_2H(X,Y) - H(P_2X, Y), \\ ('\nabla_Y F_3)X = q(Y)F_1X - p(Y)F_1X \\ \qquad +f_3H(X, Y) - H(P_3X, Y), \end{cases}$$

$$(3.5)\quad \begin{cases} ('\nabla_X t_1)\xi = A_{f_1\xi}X - P_1A_\xi X + r(X)t_2\xi \\ \qquad - q(X)t_3\xi, \\ ('\nabla_X t_2)\xi = A_{f_2\xi}X - P_2A_\xi X + p(X)t_3\xi \\ \qquad -r(X)t_1\xi, \\ ('\nabla_X t_3)\xi = A_{f_3\xi}X - P_3A_\xi X + q(X)t_1\xi \\ \qquad -p(X)t_2\xi, \\ (\nabla_X^\perp f_1)\xi = -F_1A_\xi X - H(X, t_1\xi) \\ \qquad +r(X)f_2\xi - q(X)f_3\xi, \\ (\nabla_X^\perp f_2)\xi = -F_2A_\xi X - H(X, t_2\xi) \\ \qquad +p(X)f_3\xi - r(X)f_1\xi, \\ (\nabla_X^\perp f_3)\xi = -F_3A_\xi X - H(X, t_3\xi) \\ \qquad +q(X)f_1\xi - p(X)f_2\xi. \end{cases}$$

Lemma 2 ([1]). *Let M be a quaternionic CR-submanifold of a quaternionic projective space $QP^m(c)$ with constant Q-sectional curvature $c=4$. Then M is a QR-product if and only if*

$$A_{\phi_r D^\perp}D = 0 \ (r=1,2,3).$$

Theorem 3. *Let M be a quaternionic CR-submanifold of a quaternionic projective space $QP^m(c)$ with constant Q-sectional curvature $c=4$. If the f-three-structure $\{P_1, P_2, P_3\}$ is parallel on M, then the submanifold M is a QR-product.*

Proof. From the first equation in Lemma 1, we have

$$A_{F_1X}Y + r(Y)P_2X - q(Y)P_3X + t_1H(X,Y) = 0$$

for any $X, Y \in TM$. If we take X in $D^\perp$, then $A_{F_1X}Y + t_1H(X, Y) = 0$. Using (2.7), we obtain

$$\langle A_{F_1X}Y,\ Z\rangle = \langle H(X,Y), F_1Z\rangle$$

for any tangent vector field Z. Again, putting $Z \in D$, the equation gives

$$\langle H(Y,Z), F_1X\rangle = 0,$$

which is equivalent to

$$\langle A_{F_1X}Z,\ Y\rangle = 0, \text{ i.e., } A_{\phi_1 D^\perp}D = 0.$$

In the similar way, we have $A_{\phi_r D^\perp}D = 0$. Therefore, by Lemma 2, the submanifold M is a QR-product.

Remark. *In* [5], *the one of present author and J.S. Pak proved that there does not exist proper QR-products in a quaternionic projective space if the immersion $i: M \to QP^m(c)$ is constant isotropic.*

Combining this fact with Theorem 3, we get

Theorem 4. *Let $i: M \to QP^m(c)$ be a constant isotropic immersion and M be a quaternionic CR-submanifold of $QP^m(c)$. If the f-three-structure $\{P_1, P_2, P_3\}$ on M is parallel, then the immersion i is necessarily invariant or totally real.*

Proposition 5. Let M be a submanifold of a quaternionic Kaehlerian manifold $\bar{M}$. Then F_1, F_2 and F_3 are parallel if and only if t_1, t_2 and t_3 are parallel respectively.

Proof. t_1 is parallel if and only if, for any X, Y tangent to M, ξ normal to M

$$\langle A_{f_1\xi}X,\ Y\rangle - \langle P_1A_\xi X,\ Y\rangle + \langle r(X)t_2\xi,\ Y\rangle \\ -\langle q(X)t_3\xi,\ Y\rangle = 0,$$

i.e.,

$$\langle H(X,\ Y), f_1\xi\rangle + \langle H(X,\ P_1Y), \xi\rangle \\ +\langle r(X)t_2\xi,\ Y\rangle - \langle q(X)t_3\xi,\ Y\rangle = 0,$$

which is equivalent to

$$-\langle f_1H(X,\ Y),\ \xi\rangle + \langle H(X,\ P_1Y),\ \xi\rangle \\ -\langle \xi,\ r(X)F_2Y\rangle + \langle \xi,\ q(X)F_2Y\rangle = 0$$

with the aid of (2.6) and (2.7), i.e.,

$$-f_1H(X,\ Y) + H(X,\ P_1Y) - r(X)F_2Y \\ +q(X)F_3X = 0,$$

which is equivalent to the fact that F_1 is parallel. The others are obtained by the same method.

Now, we denote by ν the subbundle of the normal bundle $T^{\perp}M$ which is the orthogonal complement of $ID^{\perp}\oplus JD^{\perp}\oplus KD^{\perp}$, i.e., $T^{\perp}M=\mu\oplus\nu$, $\langle\nu, \phi_r D^{\perp}\rangle=0$, where $\mu=\bigoplus_r \phi_r D^{\perp(1)}$.

Proposition 6. Let M be a quaternionic CR-submanifold in a quaternionic Kaehler manifold $\bar{M}$. If F_1, F_2 and F_3 are parallel, then $\langle H(X, Y), \nu\rangle=0$, for any $X, Y\in TM$.

Proof. By lemma 1, $'\nabla_x F_1=0$ if and only if, for any $X\in TM$, $Y\in D^{\perp}$,

$$r(X)F_2Y-q(X)F_3Y+f_1H(X, Y)=0.$$

Hence $f_1H(X,Y)\in\bigoplus_{r=2}^{3}\phi_r D^{\perp}$. By the quite similar way, $f_rH(X, Y)\in\bigoplus_r\phi_r D^{\perp}$ for any $X\in TM$, $Y\in D^{\perp}$. So, $\langle\phi_r H(X, Y), \nu\rangle=0$, which means that $\langle H(X, Y), \nu\rangle=0$, i.e.,

(3.6) $\langle H(D, D), \nu\rangle=0$, $\langle H(D, D^{\perp}), \nu\rangle=0$.

On the other hand, from (2.2) and (3.1), we have

$$\langle H(X, Y), \phi_r\xi\rangle=\langle\bar{\nabla}_X Y, \phi_r\xi\rangle=-\langle\bar{\nabla}_X(\phi_r Y), \xi\rangle$$
$$=-\langle H(X, \phi_r Y), \xi\rangle$$

for any vector fields X, Y in D, Z in $D^{\perp}$ and ξ in ν. Hence we have

$$\langle H(\phi_s X, \phi_r Y), \xi\rangle=\langle H(X, Y), \phi_s\phi_r\xi\rangle$$
$$=\langle H(X, Y), \phi_r\phi_s\xi\rangle, \quad r\neq s.$$

Since $\phi_s\phi_r=-\phi_r\phi_s$, this impliss $\langle H(D, D), \nu\rangle=0$. From which and (3.6), we have completed our proof.

The structures f_1, f_2 and f_3 are parallel in the normal bundle of M in $\bar{M}^{4m}$ if

$$(\nabla^{\perp}_X f_1)\xi=r(X)f_2\xi-q(X)f_3\xi,$$
$$(\nabla^{\perp}_X f_2)\xi=p(X)f_3\xi-r(X)f_1\xi,$$
$$(\nabla^{\perp}_X f_3)\xi=q(X)f_1\xi-p(X)f_2\xi,$$

(see[4],[6]).

Proposition 7. Let M be a quaternionic CR-submanifold of a quaternionic Kaehlerian manifold of $\bar{M}$. If the f-three structure is parallel in the normal bundle, then $\langle H(X, Y), \nu\rangle=0$ for any $X, Y\in TM$.

Proof. It follows from Lemma 1 and assumption that

$$F_1A_\xi X=-H(X, t_1\xi),$$

which implies

$$IA_\xi X=P_1A_\xi X-H(X, t_1\xi).$$

Applying I to the above equation, we easily obtain

$$-A_\xi X=P_1^2A_\xi X-t_1H(X, t_1\xi)-f_1H(X, t_1\xi)$$

with the aid of $F_rP_r=0(r=1,2,3)$. Comparing the tangential and normal parts of both sides, we get

$$-A_\xi X=P_1^2A_\xi X-t_1H(X, t_1\xi), \quad f_1H(X, t_1\xi)=0.$$

The last equation implies that $IH(X, t_1\xi)=t_1H(X, t_1\xi)$. It follows from which and (3.3) that

$$0=\langle IH(X, t_1\xi), \eta\rangle=-\langle H(X, t_1\xi), f_1\eta\rangle$$
$$=-\langle A_{f_1\eta}X, t_1\xi\rangle=\langle F_1A_{f_1\eta}X, \xi\rangle$$

i.e.,

$$F_1A_{f_1\eta}X=0, \quad \eta\in T^{\perp}M$$

which means that $A_{f_1\eta}X\in D$ for any $X\in TM$, $\eta\in T^{\perp}M$. Thus, it is clear that $A_{f_r\eta}X\in D$, So, $A_\nu D^{\perp}=0$. From which and the proof of Proposition 6, we have completed our proof.

References

(1) M. Barros, B.Y. Chen and F. Urbano, Quaternion CR-submanifolds of quaternion manifolds, Kodai Math. J. 4(1981)

(2) S. Ishihara, Quaternionic Kaehlerian manifolds, J. Diff. Geo., 9(1974), 483—500.

(3) B. O'Neill, Isotropic and Kaehler immersions, Can, J. Math. 17(1965), 907—915.

(4) J.S. Pak, Quaternionic CR-submanifolds of quaternionic space forms, Precceding of Symposia in pure and applied mathematics, Vol. II(1982), 270—282.

(5) J.S. Pak and T.H. Kang, Planar geodesic submanifolds in a quaternionic projective space, to appear in Geometriae Dedicata.

(6) Y. Shibuya, Real submanifolds in a quaternionic projective space, Kōdai Math. J. Vol 1, No. 3(1978), 421—439.