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A Note on Semi-Closed Graph

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⟨Abstract⟩

In this note, various conditions are discussed under which a mapping has a semi-closed graph. If a mapping has a closed graph, then, clearly, its graph is semi-closed but not conversely (Example 1). It is shown that, on one hand, various types of some known mappings may fail to have semi-closed graphs and, on the other hand, a mapping having semi-closed graph may fail to be some 'nice' types of mappings.

반폐 그래프에 관하여

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⟨요　약⟩

이 논문에서 사상이 반폐그래프를 갖기 위한 여러가지 조건들을 알아본다. 사상이 폐그래프를 가지면 그 그래프는 반폐 그래프이나, 그 역은 성립하지 않음을 보인다. 또 이미 알려진 몇 개의 사상들이 반폐 그래프를 갖지 못하는 것을 보이고, 반면, 반폐 그래프를 가져도 항상 좋은 형의 사상이 되지 않음을 보였다.

I. Preliminaries

Throughout this paper, a space means a topological space. In a space X, a set A is *semi-open* [14] iff there exists an open set O in X such that $O\subset A\subset cl\ O$, where $cl\ O$ denotes the *closure* of O in X. Every open set is semi-open [14]. $SO(X)$ will denote the class of all semi-open sets in the space X [14]. $[A\in SO(X)$ and $B\in SO(Y)]$ implies $A\times B\in SO(X\times Y)$ [14]. A subset M of a space X is a *semi-neighbourhood* [1] of a point x of X iff there exists a semi-open set A in X such that $x\in A\subset M$. A set A in X is semi-open iff it is a semi-neighbourhood of each of its points [1]. A set A in X is *semi-closed* [2] iff $(X-A)$ is semi-open. The *semi-closure* [2] of a subset A of X, denoted by $scl\ A$, is the intersection of all the semi-closed sets containing A. Note that $A\subset scl\ A\subset cl\ A$ [2], $scl(scl\ A)=scl\ A$ [2]; $A\subset B$ implies $scl\ A\subset scl\ B$; and A is semi-closed iff $A=scl\ A$ [2]. By a *semi-clopen set* A, we shall mean a set A which is both semi-open and semi-closed.

II. Mappings with Semi-Closed Graphs

If $f: X\longrightarrow Y$, then $G(f)=\{(x,f(x)) : x\in X\}$ is called the *graph of f*. If the set $G(f)$ is semi-closed in the product space $X\times Y$, then f

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is said to have a semi-closed graph. Clearly, for $f: X \longrightarrow Y$, the graph $G(f)$ is semi-closed if, for each $(x, y) \notin G(f)$, there exist semi-open sets U and V containing x and y, respectively, such that $(U \times V) \cap G(f) = \phi$.

Theorem 1: *The mapping $f: X \longrightarrow Y$ has a semi-closed graph if, for each $x \in X$, $y \in Y$ such that $y \neq f(x)$, there exist semi-open sets U and V containing x and y, respectively, such that $f(U) \cap V = \phi$.*

Proof: Let $x \in X$, $y \in Y$ such that $y \neq f(x)$. Then $(x, y) \notin G(f)$. Since semi-open sets U and V are such that $f(U) \cap V = \phi$, it follows that $U \times V$ is a semi-open set in $X \times Y$ [14, Theorem 11] such that $(U \times V) \cap G(f) = \phi$. Hence, $G(f)$ is semi-closed.

Remark 1: Each closed graph is, obviously, semi-closed. But the converse may fail to be true, this may be seen by the following example.

Example 1: Let $X = \{a, b, c\}$, and $\mathscr{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ be the topology on X. Let $i: X \longrightarrow X$ be the identity mapping. Then, obviously, $G(i)$ is semi-closed but not closed.

$f: X \longrightarrow Y$ is *set-s-connected* [4] iff, for any semi-clopen subset F of $f(X)$, $f^{-1}(F)$ is semi-clopen in X.

Remark 2: Various types of known mappings, viz., a continuous mapping a θ-continuous mapping [5], a mapping almost continuous in the sense of Frolik [6], a weakly continuous mapping [13], a semi-continuous mapping [14], a mapping almost continuous in the sense of Husain [8], a mapping almost continuous in the sense of M.K. Singal and A.R. Singal [20], a c-continuous mapping [7], a c^*-continuous mapping [18], a H-continuous mapping [15], a s-continuous mapping [10], a set-connected mapping [11], a set-s-connected mapping [4], an irresolute mapping [3], a connected mapping [19], a semi-connected mapping [12], and a weak semi-connected mapping [9] may fail to have a semi-closed graph. This may be seen by the following example:

Example 2: Let X be a space, containing more than one point, with the indiscrete topology, and let $i: X \longrightarrow X$ be the identity mapping. Here, the graph $G(i)$ is not semi-closed.

Remark 3: A mapping having a semi-closed graph may fail to be continuous, semi-continuous [14], irresolute [3], set-connected [11], set-s-connected [4], connected [19], semi-connected [12], weak semi-connected [9], s-continuous [10], H-continuous [15], c-continuous [7], c^*-continuous [18], almost continuous in the sense of Frolik [6], almost continuous in the sense of Husain [8], almost continuous in the sense of M.K. Singal and A.R. Singal [20], weakly continuous [13], θ-continuous [5]. It may be seen by the following example.

Example 3: Let $X = \{a, b, c\}$, and consider on X the topologies $\mathscr{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathscr{T}' = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then, obviously, the identity mapping $i: (X, \mathscr{T}) \longrightarrow (X, \mathscr{T}')$ has a semi-closed graph, but i is none of the mappings mentioned in Remark 3.

Definition 1: A space X is said to be *extremely s-disconnected* iff the semi-closure of every semi-open set is semi-open.

In order to say the following, we note that a space X is *semi-T_2* [17] iff to each pair of distinct points x, y of X there exist disjoint semi-open sets A and B in X such that $x \in A$ and $y \in B$.

Equivalently, we have X is *semi-T_2* iff to each pair of distinct points x, y of X there exists a semi-open set N in X containing x such that $y \notin scl\ N$.

Theorem 2: *Let $f: X \longrightarrow Y$ be a set-s-connected surjection, and Y be an extremely s-disconnected and semi-T_2 space. Then $G(f)$ is semi-closed in $X \times Y$.*

Proof: Let $(x, y) \notin G(f)$. Then $x \in X$, $y \in Y$ and $y \neq f(x)$. Since Y is *semi-T_2*, there exists a semi-open set $N \subset Y$ containing y such that $f(x) \notin scl\ N = V$. Then, Y being extremely s-

disconnected, V is a semi-clopen subset of Y and so, f being set-s-connected surjection, $f^{-1}(V)$ is semi-clopen in X and $x \notin f^{-1}(V)$. Taking $U = X - f^{-1}(V)$, U is a semi-open set containing x, and then $f(U) \cap V = \phi$. Consequently, $G(f)$ is semi-closed.

Theorem 3: *Let $f: X \longrightarrow Y$ be a semi-continuous mapping, where Y is T_2. Then $G(f)$ is semi-closed.*

Proof: Let $(x, y) \notin G(f)$. Then $x \in X$, $y \in Y$, and $y \neq f(x)$. Hence, Y being T_2, there exist disjoint open sets U and V containing $f(x)$ and y, respectively. Since f is semi-continuous, $f^{-1}(U)$ is a semi-open set containing x. Then, clearly, $f^{-1}(U) \times V$ is a semi-open subset containing (x, y) in $X \times Y$ [14, Theorem 11]. Now, let $x_1 \in f^{-1}(U)$ such that $f(x_1) \in V$. Then, obviously, we have $U \cap V \neq \phi$, a contradiction. Hence, there is no $x_1 \in f^{-1}(U)$ such that $f(x_1) \in V$. Consequently, $(f^{-1}(U) \times V) \cap G(f) = \phi$ which implies $G(f)$ is semi-closed.

Theorem 4: *Let $f: X \longrightarrow Y$ be irresolute, where Y is semi-T_2. Then $G(f)$ is semi-closed.*

The proof of the above Theorem 4, is essentially, similar to that of Theorem 3.

The following corollary is an immediate consequence of Theorem 4.

Corollary 1. [17, Theorem 5.3]: *If X is semi-T_2, then the identity mapping $i: X \longrightarrow X$ has a semi-closed graph.*

Definition 2 [16]: A space X is *locally s-connected* iff for every point $x \in X$ and every open set O containing x, there exists an open s-connected set G such that $x \in G \subset O$.

Definition 3 [16]: Two sets A and B of a space X are said to be *semi-separated* iff $A \cap scl\ B = \phi = scl\ A \cap B$.

Lemma 1 [16]: *If A and B are open, s-connected and non-semi-separated sets in the space X, then $A \cup B$ is s-connected.*

Lemma 2 [16]: *A space X is not s-connected iff it is the uinon of two non-empty disjoint semi-open sets.*

Definition 4: A mapping $f: X \longrightarrow Y$ is said to be ***almost irresolute*** on X iff for each $x \in X$ and for each semi-neighbourhood V of $f(x)$, $scl\ f^{-1}(V)$ is a semi-neighbourhood of x.

An almost irresolute mapping need not have a semi-closed graph. For, the mapping i, defined in Example 2, is almost irresolute, but $G(i)$ is not semi-closed. However, we have

Theorem 5: *Let $f: X \longrightarrow Y$ be an almost irresolute mapping and Y a locally s-connected, T_2 space. If f and f^{-1} map open s-connected sets onto open s-connected sets, then $G(f)$ is semi-closed.*

Proof: Let $(x, y) \notin G(f)$. Then $x \in X$, $y \in Y$, and $y \neq f(x)$. Since Y is a T_2 locally s-connected space, there exist open s-connected subsets U and V containing y and $f(x)$, respectively, such that $U \cap V = \phi$. Hence $f^{-1}(U) \cap f^{-1}(V) = \phi$. Then, we have $f^{-1}(U) \cap scl\ f^{-1}(V) = \phi$. For, if possible, let $f^{-1}(U) \cap scl\ f^{-1}(V) \neq \phi$. Then, by Definition 3, by hypothesis and by Lemma 1, $f^{-1}(U) \cup f^{-1}(V)$ is open s-connected, Again, by hypothesis, $f[f^{-1}(U) \cup f^{-1}(V)] = U \cup V$ is open s-connected, which is impossible in view of Lemma 2, Now, since f is almost irresolute, $scl\ f^{-1}(V)$ is a semi-neighbourhood of x and hence there exists a semi-open set $T \subset scl\ f^{-1}(V)$ containing x. Therefore, $f(T) \cap U = \phi$. Consequently, $G(f)$ is semi-closed.

Remark 4: A mapping having a semi-closed graph may fail to be almost irresolute. For, the mapping i, defined in Example 3, has a semi-closed graph but is not almost irresolute.

Definition 5 [3]: $f: X \longrightarrow Y$ is said to be a ***semi-homeomorphism*** iff f is bijective, irresolute, and pre-semi-open. X and Y are said to be ***semi-homeomorphic***.

Definition 6 [3]: A property which is preserved under semi-homeomorphisms is said to be a ***semi-topological property***.

Hence, the property of a subset of a space being semi-closed is a semi-topological property.

Theorem 6: *If $f: X \longrightarrow Y$ is bijective and* *has a semi-closed graph, then f^{-1} has a semi-closed graph.*

Proof: Let $G(f)$ and $G(f^{-1})$ denote the graphs of f and f^{-1}, respectively. Then, clearly, $G(f) = \{(x,y) : y=f(x),\ x\in X\}$, where as $G(f^{-1})= \{(y,x) : x=f^{-1}(y), y\in Y\}$. Since $f^{-1}(y)=x$ iff $y=f(x)$, it follows that $G(f)$ and $G(f^{-1})$ are semi-homeomorphic under the semi-homeomorphism $(x,y) \longrightarrow (y,x)$ of $X\times Y$ onto $Y\times X$. Since $G(f)$ is semi-closed, so is $G(f^{-1})$.

Definition 7 : A net $\{x_\alpha : \alpha \in D\}$, in a space X, ***s-converges*** to a point $x_0\in X$ iff it is eventually in every semi-neighbourhood of x_0.

Theorem 7 : *If Y is a semi-open set in a space X, then no net in $X-Y$ can s-converge to a point of Y.*

Proof: Let $\{x_\alpha : \alpha\in D\}$ be a net in X such that $\{x_\alpha : \alpha \in D\}$ s-converges to a point $x_0\in Y$. Since Y is semi-open, and so, a semi-neighbourhood of x_0, $\{x_\alpha : \alpha \in D\}$ is eventually in Y. This means that $\{x_\alpha : \alpha \in D\}$ is always in Y, that is, $\{x_\alpha : \alpha \in D\}$ is never in $X-Y$.

The following corollary is an immediate consequence of Theorem 7.

Corollary 2 : *If A is a semi-closed set in X, then no net in A can s-converge to a point of $X-A$.*

Definition 8 : If a net $\{x_\alpha : \alpha \in D\}$, in space X, s-converges to a point $x_0 \in X$, then we say that x_0 is a ***semi-limit point*** of $\{x_\alpha : \alpha \in D\}$. Symbolically, we write $x_0 \in$ *semi-limit*$_\alpha$ (x_α).

Theorem 8 : *Let $f : X\longrightarrow Y$ be a mapping. If $G(f)$ is semi-closed and if, for each net $\{x_\alpha : \alpha \in D\}$ in X s-converging to $x \in X$, the net $\{f(x_\alpha) : \alpha \in D\}$ s-converges to $y \in Y$, then $y=f(x)$.*

Proof: Let $\{x_\alpha : \alpha \in D\}$ be a net in X s-converging to $x \in X$ such that $\{f(x_\alpha) : \alpha \in D\}$ s-converges to $y \in Y$. Then, clearly, $(x_\alpha, f(x_\alpha))\in G(f)$ for all $\alpha \in D$. Since $G(f)$ is semi-closed, by Corollary 2 and Definition 8, *semi-limit*$_\alpha(x_\alpha, f(x_\alpha)) \in G(f)$. But *semi-limit*$_\alpha(x_\alpha, f(x_\alpha))=(x,y)$. Hence $(x,y)\in G(f)$ which implies $y=f(x)$.

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