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Properties of s-connected spaces

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〈Abstract〉

Two subsets A and B of a topological space X are said to be *semiseparated* iff $A \cap \mathrm{scl}(B)=\phi=\mathrm{scl}(A) \cap B$ and X is said to be *s-connected* iff it can not be expressed as a union of two nonempty semiseparated sets. The present paper will give some properties and characterizations of s-connected spaces and some results in [7] are improved.

s-연결 공간의 성질들에 관하여

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〈요 약〉

이 논문에서는 s-연결 공간의 여러가지 성질을 알아보고 논문 [7]의 몇 가지 결과들을 일반화한다.

I. Introduction

S. N. Maheshwari[7] defined that two subsets A and B of a topological space X are said to be *semiseparated* if and only if $A \cap \mathrm{scl}(B)=\phi=\mathrm{scl}(A) \cap B$ and X is said to be *s-connected* if and only if it can not be expressed as a union of two nonempty semiseparated sets. The present paper will give some properties and characterizations of s-connected spaces and some results in [7] are improved.

Throughout this note, spaces mean topological spaces, iff means if and only if, and X will be a space equipped with a topology $T(X)$ without specific mentions. Let A be a subset of X. The closure and the interior of A in X will be denoted, respectively, by $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ ($\mathrm{cl}(A)$ and $\mathrm{int}(A)$ without confusions). A is said to be *semiopen* iff $A \subset \mathrm{cl}(\mathrm{int}(A))$, *semiclosed* iff $\mathrm{int}(\mathrm{cl}(A)) \subset A$, *feebly open*[8] (equivalent to *α-set*[1]) iff $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, *feebly closed* (equivalent to *coα-set*) iff $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subset A$, *preopen*[9] iff $A \subset \mathrm{int}(\mathrm{cl}(A))$, *preclosed* iff $\mathrm{cl}(\mathrm{int}(A)) \subset A$, and *regular semiopen*[6] iff it is semiopen and semiclosed. The interseciton of all semiclosed sets containing A is called the semiclosure of A and denoted by $\mathrm{scl}(A)$. $FO(X)$, $PO(X)$ and $SO(X)$ will be denoted, respectively, by the families of all feebly open, preopen and will semiopen sets in a space X. By $f: X \to Y$, we denote a function from a space X into a space Y. A function $f: X \to Y$ is said to be *semi continuous* [5] iff for each $V \in T(Y)$, $f^{-1}(V)$

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$\in SO(X)$, and ***preopen*** [9] if for each $U\in T(X)$, $f(U)\in PO(Y)$. Every continuous (resp. open) function is semicontinuous (resp. preopen) but the converses may not be true.

II. Preliminaries

We will meet almost all properties in this section in [1, 7, 10, 12, 13], which will need in section 3.

The implication $T(X)\Rightarrow FO(X)\Rightarrow PO(X)$ holds for any space X and their converse may not, in general, be true. It was shown in [7] that every s-connected space is connected but the converse may be false. A subset A of X is s-connected iff it is an s-connected subspace of X. It is easy to show that C and D are semiseparated if $C\subset A$ and $D\subset B$, where A and B are semiseparated sets.

LEMMA 2.1. *A space X is s-connected iff nononempty proper subset of X is both semiopen and semiclosed* [7].

It follows immediately from Lemma 2.1 that a space X is not s-connected iff it is a union of two nonempty disjoint semiopen (or semiclosed) sets. It is shown that if $A\subset Y\subset X$ and $A\in PO(X)$, then $A\in PO(Y)$, and that the intersection of a feebly open set and a semiopen set is semiopen.

LEMMA 2.2. *If* $A\in PO(X)$ *and* $V\in SO(X)$, *then* $A\cap V\in SO(A)$[10].

LEMMA 2.3. *If* $A\subset Y\subset X$ *and* $Y\in PO(X)$, *then* $\mathrm{scl}_X(A)\cap Y=\mathrm{scl}_Y(A)$[13].

The proofs of following lemmas are easy and are thus omitted.

LEMMA 2.4. *If* $A\in RO(X)$ *and* $A\subset B\subset \mathrm{int}(\mathrm{cl}(A))$, *then* $B\in PO(X)$.

LEMMA 2.5. $\mathrm{scl}(A)=A\cup \mathrm{int}(\mathrm{cl}(A))$ *for each subset A of X.*

It follows immediately from Lemma 2.5 that $\mathrm{int}(\mathrm{cl}(A))\subset \mathrm{scl}(A)$ for each subset A of X.

LEMMA 2.6. *If* $f:X\to Y$ *is preopen and semicontinuous, than* $f^{-1}(B)\in SO(X)$ *for each* $B\in SO(Y)$ [10].

III. Properties of s-connected spaces

THEOREM 3.1. *If A and B are semiseparated and* $A\cap B\in FO(X)$, *then* $A\in SO(X)$ *and* $B\in SO(X)$.

Proof: Since $A\cap \mathrm{scl}(B)=\phi=\mathrm{scl}(A)\cap B$, we have $(A\cup B)\cap(X-\mathrm{scl}(B))=A$. Because $A\cup B\in FO)X)$ and $X-\mathrm{scl}(B)\in SO(X)$, the proof is complete.

COROLLARY 3.2. *If A and B are semiseparated and* $A\cup B\in T(X)$, *then* $A\in SO(X)$ *and* $B\in SO(X)$[7].

In Theorem 3.1, $A\cup B\in FO(X)$ can not be replaced by $A\cup B\in PO(X)$, or $A\cup B\in SO(X)$ as shown by the following examples.

EXAMPLE 3.3. Let $X=\{a,b,c,d\}$ and $T(X)=\{\phi,\{a\},\{b\},\{a,b\},X\}$. Then $\{b,c\}$ and $\{d\}$ are semiseparated and $\{b,c,d\}\in SO(X)$ but $\{d\}\notin SO(X)$.

EXAMPLE 3.4. Let $X=\{a,b,c\}$ and $T(X)=\{\phi,\{a\},\{b,c\},X\}$. Then $\{a\}$ and $\{b\}$ are semiseparated and $\{a\}\cup\{b\}\in PO(X)$ but $\{b\}\notin SO(X)$.

THEOREM 3.5. *If A and B are semiseparated and* $A\cup B\in PO(X)$, *then* $A\in SO(A\cup B)$ *and* $B\in SO(A\cup B)$.

Proof: Utilize Lemma 2.2.

COROLLARY 3.6. *Let A and B be nonempty semiseparated sets of X and* $A\cup B\in PO(X)$ *(thus* $A\cup B\in FO(X)$, $A\cup B\in T(X)$*), then* $A\cup B$ *is not s-connected.*

THEOREM 3.7. *Let Y be a subspace of X and A and B subsets of Y. If* $Y\in PO(X)$ *(thus* $Y\in FO(X)$, $Y\in T(X)$*), then A and B are semiseparated in X iff they are semiseparated in Y.*

Proof: It follows from Lemma 2.3.

In Theorem 3.7, $Y\in PO(X)$ can not replaced by $Y\in SO(X)$ because, let $Y=\{b,c,d\}$ be a subspace of X in Example 3.3. Then $Y\in SO(X)$. However, $\{b,c\}$ and $\{d\}$ are semiseparated in X but not semiseparated in Y.

THEOREM 3.8. *A space X is s-connected iff no nonempty proper subset of X is regular semiopen.*

Proof: It follows from Lemma 2.1 and definition of a regular semiopen set.

THEOREM 3.9. *If E is preopen, s-connected and A and B are semiseparated in X such that* $E\subset A\cup B$, *then either* $E\subset A$ *or* $E\subset B$.

Proof: The proof is similar to the case of the connectedness and is thus omitted.

COROLLARY 3.10. *If E is an s-connected, feebly open (thus, open set) and A and B are semiseparated in X such that* $E\subset A\cup B$, *then either* $E\subset A$ *or* $E\subset B$.

THEOREM 3.11. *Let* $E\subset F\subset X$ *and E and F are preopen. If E is s-connected in X, then E is also s-connected in F.*

Proof: Suppose that E is not s-connected in F. That is there are nonempty semiseparated sets A and B in the subspace E of F such that $A\cup B=E$. Since $E\in PO(X)$ and $E\subset F$, $E\in PO(F)$ from [12]. Therefore, A and B semiseparated in F, by Theorem 3.7, and since $F\in PO(X)$, they are semiseparated in X. It follows from Theorem 3.5 that A and B are semiopen in the subspace E of X. Evidently $A\cap B=\phi$. Hence E is not s-connected in X. We have a contradiction. Hence E is s-connected in F.

COROLLARY 3.12. *Let* $E\subset F\subset X$ *and E and F be feebly open (thus, open), then so it is in F if E is s-connected in X.*

THEOREM 3.13. *Let E be a preopen, s-connected set and* $E\subset F\subset \mathrm{int}(cl(E))$, *then F is s-connected.*

Proof: Assume that F is not s-connected. Then there are nonempty sets A and B which are semiseparated in F such that $F=A\cup B$. Since $E\in PO(X)$ and $E\subset F\subset \mathrm{int}(\mathrm{scl}(E))$, $F\in PO(X)$ by Lemma 2.8. Therefore, by Theorem 3.11, E is s-connected in F and $E\in PO(F)$, and so either $E\subset A$ or $E\subset B$. Let $E\subset A$. By Lemma 2.5, $B\subset \mathrm{int}(\mathrm{scl}(E))\subset \mathrm{scl}(E)$ and $\mathrm{scl}(E)\subset \mathrm{scl}(A)$. It follows from Lemma 2.3 that $B=\mathrm{scl}(E)\cap B\subset \mathrm{scl}(A)\cap F\cap B\subset \mathrm{scl}_F(A)\cap B=\phi$ since A and B are semiseparated in F. Thus $B=\phi$, a contradiction. So F is s-connected.

The next theorems 3.14, 3.15 and 3.16 are generalizations of theorem 13, 14 and corollary 2 of [7], respectively.

THEOREM3.14. *If A and B are preopen, s-connected and nonsemiseparated sets in X, then* $A\cup B$ *is s-connected.*

Proof: Suppose not. Then there are nonempty and semiseparated set C and D in $A\cup B$ such that $A\cup B=C\cup D$. Since $A\cup B\in PO(X)$, C and D are semiseparated in X, and $A\cap C$ and $A\cap D$ are semiseparated in A by Theorem 3.7. Since either $A\cap D=\phi$ or $A\cap C=\phi$ from $A=(A\cap C)\cup(A\cap D)$, we have either $A\subset C$ or $A\subset D$.

In the same way, either $B\subset C$ or $B\subset D$. If $A\subset C$ and $B\subset C$, then $A\cup B\subset C$, so that $C\cup D\subset C$ which implies that $D=\phi$. We have a contradiction. Thus $A\subset C$ implies $B\subset D$. Similarly, $A\subset D$ implies $B\subset D$. Consequently A and B are semiseparated in X. This is a contradiction. Hence $A\cup B$ is s-connected.

THEOREM 3.15. *If* $\{D_\lambda \mid \lambda\varepsilon\Lambda\}$ *is a family of preopen s-connected sets such that one of them,* $D_{\lambda 0}$, *is not semiseparated from each other member, then* $\cup D_\lambda$ *is s-connected.*

Proof: Let $E=\cup D_\lambda$. Assume that E is not s-connected. Then there are nonempty sets A and B being semiseparated in X such that $E=A\cup B$. Since $EP\in O(X)$, A and B are semiseparated in X by theorem 3.7. Now $D_{\lambda 0}\subset A\cup B$, therefore, by theorem 3.9, either $D_{\lambda 0}\subset A$ or $D_{\lambda 0}\subset B$. Let $D_{\lambda 0}\subset A$. Since $D_{\lambda 0}$ and D_λ are not semiseparated for any λ, $D_{\lambda 0}\cup D_\lambda$ are preopen, s-connetced by Theorem 3.14. We assert that for each λ, $D_{\lambda 0}\cup D_\lambda\subset A$. If for some $\lambda=\lambda_\beta$, $D_{\lambda 0}\cup D_{\lambda_\beta}\subset B$, then $D_{\lambda 0}$ and D_{λ_β} are subsets of B imply that $D_{\lambda 0}$ and D_{λ_β} are semiseparated, which is absurd. Hence we have $E\subset A$. Therefore, $B=\phi$, a contradiction. Accordingly, E is s-connected.

THEOREM 3.16. *If $\{D_\lambda|\lambda\varepsilon\Lambda\}$ is a nonempty family of preopen s-connected sets such that $\cap D_\lambda\neq\phi$, then $\cup D_\lambda$ is s-connected.*

Proof: It follows from Theorem 3.15 and the fact that any two nondisjoint sets are nonsemiseparated.

THEOREM 3.17. *Let $f:X\to Y$ be onto semicontinuos and preopen. If X is s-connected, then Y is s-connected.*

Proof: If Y is not s-connected, then Y can be expressed as a union of two nonempty disjoint semiopen sets A and B. Thus, by Lemma 2.6, $f^{-1}(A)$ and $f^{-1}(B)$ are semiopen, nonempty, disjoint and their union is X. So X is not s-connected.

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