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On Compact Hypersurface of an even-dimensional Sphere

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〈**Abstract**〉

Let M be a compact orientable hypersurface of an even dimensional sphere. Then the following statements are equivalent;

(1) $\alpha^2+\beta^2+\lambda^2=1$,

(2) U is a parallel vector field,

(3) $h(X,Y)=\frac{\lambda}{\beta}g(X,Y)$.

偶數次元 球面의 完閉 可符號인 超曲面에 關하여

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(1981. 12. 30 접수)

〈要　　約〉

偶數次元 球面의 完閉 可符號인 超曲面에서 다음 命題가 同値이다.

(1) $\alpha^2+\beta^2+\lambda^2=1$.

(2) U는 平行 vector 場이다.

(3) $h(X,Y)=\frac{\lambda}{\beta}g(X,Y)$.

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0. Introduction

Many authors started to study the submanifolds of a Kahlerian manifold, specially, hypersurfaces of a sphere or a torus ([7]∼[20]).

An orientable differentiable submanifold M^{2n} of codimension 2 of a Kahlerian manifold admits what we call an $(f,g,u,v,,\lambda)$-structure ([21]∼[35]).

In 1973, Ki, Pak and Suh have studied hypersurfaces of a manifold with (f,g,u,v,λ) structures. These submanifolds admit under certain conditions what we call an $(f,g,U_{(k)},\alpha_{(k)})$-structure.

There are many papers on $(f,g,U_{(k)},\alpha_{(k)})$-structure ([36]∼[41]).

In [15], the present author studied the condition for a complete and connected hypersurface of an even-dimensional sphere to be congruent to an odd-dimensional sphere.

The main purpose of the present paper is to study the equivalent conditions on compact orientable hypersurface of an even-dimensional sphere.

I. Hypersurfaces of an even-dimensional sphere

Let S^{2n} be an even-dimensional sphere of radius 1. We consider a $(2n\text{-}1)$-dimensional manifold M covered by a system of coordinate neighborhoods $\{U;\ x^a\}$, where here and in the sequel the indices $a, b, c, d, e, \cdots$, run over the range $\{1, 2, \cdots, 2n\text{-}1\}$, and, which is differentiably immersed in S^{2n} as a hypersurface.

Since S^{2n} has a normal (f, g, u, v, λ)-structure natually induced from the flat Kahler structure of $(2n+2)$-dimensional Euclidean space E^{2n+2}, there exists the Riemannian metric g_{cb} induced on M from that of S^{2n}, a tensor field of type (1,1) $f_b{}^a$, 1-form w_b, vector fields u^e and v^e, functions α and β such that

(1.1) $f_b{}^e f_e{}^a = -\delta_b^a + u_b u^a + v_b v^a + w_b w^a$,

(1.2) $f_e{}^a u^e = -\lambda v^a + \beta w^a$,

(1.3) $f_e{}^a v^e = \lambda u^a + \alpha w^a$,

(1.4) $f_b{}^e w_e = \beta u_b + \alpha v_b$,

(1.5) $u_e u^e = 1 - \beta^2 - \lambda^2$,

(1.6) $v_e v^e = 1 - \alpha^2 - \lambda^2$,

(1.7) $w_e w^e = 1 - \alpha^2 - \beta^2$,

(1.8) $u_e v^e = -\alpha\beta$,

(1.9) $u_e w^e = -\alpha\lambda$,

(1.10) $v_e w^e = \beta\lambda$,

(1.11) $g_{ea} f_c{}^e f_b{}^a = g_{cb} - u_c u_b - v_c v_b - w_c w_b$,

that is, M admits an $(f, g, u_{(k)}, \alpha_{(k)})$*-structure.* If we put $f_{cb} = f_c{}^e g_{eb}$, then f_{cb} is skew symmetric. Denoting by ∇_c the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_c{}^a{}_b\}$ formed with g_{cb}, we have

(1.12) $\nabla_c h_{ba} - \nabla_b h_{ca} = 0$,

(1.13) $\nabla_c f_b{}^a = -g_{cb} v^a + \delta_c^a v_b - h_{ca} w^a + h_c{}^a w_b$,

(1.14) $\nabla_c w_b = -\alpha g_{cb} - h_{ce} f_b{}^e$,

(1.15) $\nabla_c u_b = -\lambda g_{cb} + \beta h_{cb}$,

(1.16) $\nabla_c v_b - \alpha h_{cb} + f_{cb}$,

(1.17) $\nabla_c \alpha = -h_{ce} v^e + w_c$,

(1.18) $\nabla_c \beta = -h_{ce} u^e$,

(1.19) $\nabla_c \lambda = u_c$,

where h_{cb} is the second fundamental tensor ([13], [36], [37]).

II. Theorems on M

In this section, we introduce some theorems on hypersurfaces of an even-dimensional sphere in order to prove the following theorems 3.1 and 3.2.

Theorem 2.1. *In a manifold with $(f, U_{(k)}, u_{(k)}, v_{(k)})$-structure, the vectors U, V and W (or the covectors u, v and w) are linearly dependent if and only if $\alpha^2 + \beta^2 + \gamma^2 = 1$* ([39]).

Theorem 2.2. *Let M be a hypersurface of $2n$-dimensional manifold $\widetilde{M}$ with normal (f, g, u, v, λ)-structure such that the function $\lambda(1-\lambda^2)$ is not zero almost everywhere on $\widetilde{M}$ and satisfies $\nabla_j u_i - \nabla_i u_j = 2f_{ji}$ (or equivalently $\nabla_j u_i + \nabla_i u_j = -2\lambda g_{ji}$). Then, the four conditions*

$S_{cb}{}^1 = (h_{ce}u^e)w_b - (h_{be}u^e)w_c + (u_c v_b - u_b v_c) = 0$,

$S_{c1}{}^a = \beta(f_c{}^e h_e{}^a - h_c{}^e f_e{}^a) - \phi(v_c v^a + w_c w^a) - w_c(h_e{}^a u^e) - v_c u^a = 0$, $\alpha^2 + \beta^2 + \lambda^2 = 1$, and

λ is a constant,

are equivalent to each other ([36]).

Theorem 2.3. *Let M be a hypersurface of a $2n$-dimensional sphere S^{2n}. Then the necessary and sufficient condition that the induced $(f, g, u_{(k)}, \alpha_{(k)})$-structure on M is normal is*

$f_j{}^t h_t{}^h - h_j{}^t f_t{}^h = 0$,

which is equivalent to

$h_{jt} f_i{}^t + h_{it} f_j{}^t = 0$ ([37]).

Theorem 2.4. *In a manifold with $(f, g, u_{(k)}, \alpha_{(k)})$-structure, the vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly independent if and only*

$1 - \alpha^2 - \beta^2 - \lambda^2 \neq 0$.

Moreover, if vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly dependent, then $h_{ji} = (\lambda/\beta)\, g_{ji}$ in M ([37]).

Theorem 2.5. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. In order for the set (f, g, p), p being given by $p^h - \lambda u^h + \mu v^h - \nu w^h$, to define an* *almost contact metric structure, it is necessary and sufficient that* $\lambda^2+\mu^2+\nu^2=1$ ([38]).

Theorem 2.6. *Let M be a complete and connected hypersurface of an even-dimensional sphere* S^{2n}. If $\alpha^2+\beta^2+\lambda^2=1$ *holds at each point of M, then M is, provided* $n>1$, *congruent to an odd-dimensional sphere* S^{2n-1} *of radius* $|\beta|$ ([15]).

By the famous Theorem of Green, we have

Theorem 2.7. *In a compact orientable Riemannian space M, we have*

$$\int_M (g^{ji}\nabla_j\nabla_i f)d\sigma=0$$

for any scalar field f, where $d\sigma$ *is the volume element*

$$d\sigma=\sqrt{g}\,d\xi^1\Lambda d\xi^2\Lambda\cdots\Lambda d\xi^n$$

([1], [2], [3], [4], [5], [6]).

III. The main Theorem

In this section, we consider that M is a compact orientable hypersurface of S^{2n}. The main theorem that the author represents on this paper is as follows;

Theorem 3.1. *Let M be a compact orientable hypersurface of an even-dimensional sphere* S^{2n}. *Then the three conditions*

(3.1) $h_{cb}=\dfrac{\lambda}{\beta}g_{cb}$,

(3.2) u^a *is a parallel vector field and*

(3.3) $\alpha^2+\beta^2+\lambda^2=1$

are equivalent to each other.

Proof. (3.1)⇒(3.2)

If $h_{cb}=\dfrac{\lambda}{\beta}g_{cb}$, then, from (1.15), we have

$$\nabla_c u_b=0,$$

i.e., u^a is a parallel vector field.

(3.2)⇒(3.1)

If $\nabla_c u_b=0$, then

$$\nabla_b\lambda^2-2\lambda\nabla_b\lambda-2\lambda u_b,$$

$$\frac{1}{2}\nabla_c\nabla_b\lambda^2=(\nabla_c\lambda)u_b+\lambda\nabla_c u_b=u_c u_b$$

by virtue of (1.19).

Moreover, from (1.5), we get

$$\frac{1}{2}g^{cb}\nabla_c\nabla_b\lambda^2=u_c u^c=1-\beta^2-\lambda^2.$$

Since M is compact, by theorem 2.7, we find

$$\frac{1}{2}\int_M g^{cb}\nabla_c\nabla_b\lambda^2 d\sigma=\int_M(1-\beta^2-\lambda^2)d\sigma=0.$$

By the fact that $1-\beta^2-\lambda^2$ is non-negative function on M, we obtain

(3.4) $1-\beta^2-\lambda^2=0$.

If we put $M_0=\{P\in M\,|\,\beta(P)=0\}$, then

$$\nabla_c u_b=-\lambda g_{cb}+\beta h_{cb}=-\lambda g_{cb}=0$$

on M_0 by virtue of (1.15) and assumption.

This equation shows that $\lambda=0$ because of

$$g_{cb}g^{cb}=2n-1\neq 0.$$

This is contrary to (3.4).

Thus M_0 is void. Therefore we get

$$\beta(P)\neq 0 \text{ on } M.$$

Hence we have

$$h_{cb}=\frac{\lambda}{\beta}g_{cb}.$$

(3.1)⇒(3.3)

i) Using (1.12) and (3.1), we find the fact that

(3.5) $\dfrac{\lambda}{\beta}$ is a constant.

From (1.17), (3.1), (3.5), (1.16), (1.14), (1.6), (1.10), (1.7), (3.4), we obtain

$$\frac{1}{2}\int_M g^{cb}\nabla_c\nabla_b\alpha^2 d\sigma=-2n\left\{\left(\frac{\lambda}{\beta}\right)^2+1\right\}\int_M\alpha^2 d\sigma=0,$$

which shows that

$$\alpha^2=0.$$

Hence, we have (3.3) because of (3.4).

ii) Or from (3.4) and (1.5), we get $u_c=0$ from which, using (1.19), we find

λ is a constant

which is equivalent to (3.3)

by virtue of theorem 2.2.

(3.3)⇒(3.1)

It is clear from Theorems 2.1 and 2.4.

(Q.E.D.)

Taking account of Theorems 2.1, 2.2, 2.5, 3.1 and 2.6, we get

Theorem 3.2. *Let M be a complete connected compact and orientable hypersurace of an even-dimensional sphere* S^{2n}. *If one of the following conditions holds;*

(1) *the vectors U, V and W are linearly dependent,*

(2) $(h_{ce}u^e)w_b-(h_{be}u^e)w_c-(u_cv_b-u_bv_c)=0$,

(3) $\beta(f_c^eh_e^a-h_c^ef_e^a)-w_c(h_e^au^e)-v_cu^a=0$,

(4) $\alpha^2+\beta^2+\lambda^2=1$,

(5) λ *is a constant,*

(6) *the set* (f, g, p), p *being given by* $p^a=-\alpha u^a+\beta v^a+\lambda w^a$, *defines an almost contact metric structure,*

(7) $h_{cb}=-\dfrac{\lambda}{\beta}g_{cb}$, *and*

(8) u^a *is a parallel vector field,*

then M is, provided $n>1$, *congruent to an odd-dimensional sphere* S^{2n-1} *of radius* $|\beta|$.

Note that $\alpha=0$ on M ([37]).

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