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RADII FOR NUMEROID AND NUMERALOID OPERATORS

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Abstract

We show the relation among numeroid, numeraloid and quasi-numeroid operator and characterize them in the Hilbert space.

Numeroid 와 Numeraloid 작용소에 대한 반경

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<요 약>

Hilbert 공간에서 Numeroid, Numeraloid와 quasi-Numeroid 작용소들의 관계를 보이고 그들을 특성화 해 보였다.

1. Introduction

Throughout this paper, H is a complex Hilbert space. For a bounded linear operator T on H, let $\sigma(T)$ denote its spectrum, $\gamma(T)$ its spectral radius, $W(T)$ its numerical range, and $\omega(T)$ its numerical radius. Let R_T and W_T (resp. z_T and n_T) be the radius (resp. center) of the smallest closed disc containing $\sigma(T)$ and $W(T)$ of T respectively. Björck and Thomee [1] introduced the transcendental radius M_T for an operator T on H as $M_T = \sup_{\|x\|=1} \sqrt{\|Tx\|^2 - |(Tx,x)|^2}$.

Björck and Thomee proved that the equality $M_T = R_T$ holds for a normal operator. Moreover, Istratescu [6] showed $M_T = R_T$ for a transloid operator. But Sheth [9] observed that the equality $M_T = R_T$ does not hold for a convexoid operator. Hence it is obvious that operators satisfying the equality $M_T = R_T$ form a class including all transloid operators. Fujii and Prasanna

[2] showed that the transcendental disc centered Stampfli's center m_T of mass with radius M_T contains $W(T)$ and $\sigma(T)$ by Garske's theorem [5], so that $R_T \leq W_T \leq M_T$. In [3], Stampfli's center m_T of mass of an operator T is a scalar defined by $\|T - m_T\|^2 + |\alpha|^2 \leq \|T - m_T + \alpha\|^2$ for all scalars α, that is, $M_T = \|T - m_T\| = \min\{\|T - \alpha\| : \alpha \in C\}$.

Fujii and Seo [3] showed the followings: (1) Let n_T define the numerical center of mass for an operator T by $\omega(T - n_T)^2 + |\alpha|^2 \leq \omega(T - n_T + \alpha)^2$ for all scalars α. Then n_T is exactly the center of the smallest closed disk containing $W(T)$ of T, that is, $\omega(T - n_T) = \min\{\omega(T - \alpha) : \alpha \in C\}$. (2) Let z_T define as the scalar satisfying $\gamma(T - z_T)^2 + |\alpha|^2 \leq \gamma(T - z_T + \alpha)^2$ for all scalars α. Then z_T is the center of the smallest closed disc containing $\sigma(T)$ of T, that is , $\gamma(T - z_T) = \min\{\gamma(T - \alpha) : \alpha \in C\}$.

The maximal numerical range of T is the subset of C such that $W_0(T) = \{\lambda \in C : (Tx_n, x_n) \to \lambda \text{ for } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}$. Stampfli [10] proved that $z = m_T$ if and only if $0 \in W_0(T - z)$. In [4] T is called normaloid, iff $\gamma(T) = \omega(T) = \|T\|$, spectraloid iff $\gamma(T) = \omega(T)$, convexoid iff $\overline{W(T)} =$ conv $\sigma(T)$ ($\overline{X}$ denotes the closure of the set $X \subseteq C$ and conv X its convex hull), transloid iff $T - \lambda I$ is normaloid for any $\lambda \in C$, centraloid iff $T - z_T$ is spectraloid and centroid iff $T - z_T$ is normaloid.

In this paper, we shall introduce numeroid,numeraloid, quasi-numeroid operator and characterize them. Moreover we shall show the equivalence relations among numeroid operator, spectral set, and strong normal dilation.

2. Characterization of numeroid and numeraloid operator

We shall introduce numeroid, numeraloid and quasi-numeroid operator as follows : T is called numeroid iff $T - n_T$ is normaloid , or equivalently iff $\overline{W(T)}$ is a spectral set, numeraloid iff $T - n_T$ is a spectraloid operator and quasi-numeroid iff T is the operator satisfying $n_T = m_T$. We shall show the relations among radii (R_T, W_T, and M_T), numeroid, numeraloid and quasi-numeroid and so characterize them.

The following lemmas are foundmental.

Lemma 2.1 .[8]. If $z \neq z_T$, then $\|T - z\| \neq R_T$.

Lemma 2.2. If $z \neq n_T$, then $\|T - z\| \neq W_T$.

Proof. Assume $n_T = 0$ without loss of generality. Then W_T is the numerical radius $\omega(T)$ of T. Suppose that $\|T - z\| = W_T$ for some $z \neq 0$. Then we have $\omega(T - z) \leq W_T = \omega(T)$, which is impossible since W_T is the radius of the smallest closed disk containing $W(T)$. Thus $\|T - z\| \neq W_T$.

Lemma 2.3. For any bounded linear operator T, $\gamma(T-z_T)=R_T$ and $\omega(T-n_T)=W_T$.

Proof. By Lemma 2.1, $|z_T-\lambda|=R_T$ for some $\lambda\in\sigma(T)$. Since $|z_T-\lambda|\le R_T$ for all $\lambda\in\sigma(T)$, R_T is the radius of the smallest closed disk containing $\sigma(T)$, we have the equality $\gamma(T-z_T)=R_T$ by the definition of the spectral radius. By Lemma 2.2, $|n_T-\alpha|=W_T$ for some $\alpha\in W(T)$. Since $|n_T-\alpha|\le W_T$ for all $\alpha\in W(T)$, we have also the equality $\omega(T-n_T)=W_T$.

We have the following characterization of numeroid operators

Theorem 2.4. An operator T is a numeroid operator if and only if $M_T=W_T$.

Proof. By Lemma 2.3, we have $R_T=\gamma(T-z_T)$ and $W_T=\omega(T-n_T)$. If T is a
numeroid operator, then $T-n_T$ is normaloid, so that $\gamma(T-n_T)=\|T-n_T\|$. It is already observed that $\|Tx\|^2-|(Tx,x)|^2=\|(T-z)x\|^2-|((T-z)x,z|^2$ for all $z\in C$. Thus we have $M_T^2=\sup_{\|x\|=1}\{\|(T-n_T)x\|^2-|((T-n_T)x,x)|^2\}\le\sup_{\|x\|=1}\{(\|T-n_T)x\|^2\}=\|T-n_T\|^2=\omega(T-n_T)^2=W_T^2$. So $M_T\le W_T$. Conversely, if $M_T=W_T$, then $W_T=\omega(T-n_T)=\min\{\omega(T-z):z\in C\}\le\omega(T-m_T)\le\|T-m_T\|=M_T$, so that $n_T=m_T$ by the uniqueness of the smallest closed disk containing $W(T)$. Thus $T-n_T$ is normaloid.

Corollary 2.5. If $W_T=R_T$, then T is a numeraloid operator.

Proof. If $W_T=R_T$,then $\gamma(T-n_T)\le\omega(T-n_T)=W_T=R_T$, , so that $z_T=n_T$ by the uniqueness of the smallest closed disc containing $\sigma(T)$. Thus $T-n_T$ is a spectraloid operator.

Example 1. There is a non-numeroid numeraloid operator.

Let $T=\begin{pmatrix}0&2\\0&0\end{pmatrix}\oplus\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Then $\sigma(T)=\{-1,0,1\}$ and $\overline{W(T)}$ is the closed unit disc. Hence we have $W_T=R_T=1$, so that T is a numeraloid operator. On the other hand we have $m_T=0$ and so $M_T=\|T\|=2$. Hence T is not a numeroid operator.

From Theorem 2.4 and Corollary 2.5, the following is easily proved

Corollary 2.6. The class of operators satisfying $M_T=R_T$ includes the class of transloid operators.

Prasanna [8] showed that the class of centroid operators contains properly the class of transloid operators. Also it is observed that the class of numeroid operators contains properly the class of transloid operators. Prasanna [8] showed that the class of centroid operators is different from the class of convexoid operators.

Similarly we have the following:

Example 2. The class of numeroid operators differs from the class of convexoid operators

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and let B be the normal operator with $\sigma(B) = \{-1, 1\}$. Put $T = A \oplus B$. Then $\sigma(A) = \{0\}$ and $\overline{W(A)} = \{\lambda \in C : |\lambda| \le \frac{1}{2}\}$. Since B is normal, we have $\overline{W(B)} = \text{conv } \sigma(B) = [-1, 1]$, that is a convexoid operator B. Since $\overline{W(T)} = \text{conv } (\overline{W(A)} \cup \overline{W(B)}) \ne [-1, 1] = \text{conv } \sigma(T)$, $T = A \oplus B$ is not convexoid. whilst $n_T = 0$, $W_T = M_T = 1$, and so T is a numeroid operator by Theorem 2.4.

We shall give the characterization of quasi-numeroid operators. Stampfli [10] showed the following Lemma.

Lemma 2.7. If $\|T\| \le \|T + \lambda\|$ for all $\lambda \in C$, then $0 \in W_0(T)$.

The following theorem is parallel characterization of quasi-numeroid operators corresponding to Theorem 2.4

Theorem 2.8. $n_T = m_T$ if and only if $0 \in W_0(T - n_T)$.

Proof. If the equality $n_T = m_T$ (i.e. T is a quasi-numeroid operator) holds, we have the equalities $M_T = \|T - m_T\| = \|T - n_T\|$. Since m_T is the Stampfli's center of mass of T, it follows from the fact that $\|T - m_T\|^2 + |\alpha|^2 \le \|T - m_T + \alpha\|^2$ for all $\alpha \in C$, that $\|T - m_T\| \le \|T - m_T + \alpha\|$ for all $\alpha \in C$. Then $0 \in W_0(T - m_T) = W_0(T - n_T)$ by Lemma 2.7. Conversely, if $0 \in W_0(T - n_T)$, there exists a sequence $\{x_n\}$ such that $\|x_n\| = 1$, $((T - n_T)x_n, x_n) \to 0$, and $\|(T - n_T)x_n\| \to \|T - n_T\|$ as $n \to \infty$. Thus, it follows that $\lim_{n\to\infty} \{\|(T - n_T)x_n\|^2 - |((T - n_T)x_n, x_n)|^2\} = \|T - n_T\|$, we have the equality $\|T - m_T\| = \|T - n_T\|$, so that $n_T = m_T$ by the uniqueness of the smallest closed disc containing $W(T)$. Therefore T is a quasi-numeroid operator.

Corollary 2.9. T is a quasi-numeroid operator if and only if $M_T = \|T - n_T\|$.

Proof. If T is quasi-numeroid, $n_T = m_T$ by definition of the quasi-numeroid operator. Hence we have the equalities $M_T = \|T - m_T\| = \|T -$

$n_T\|$. Conversely, if $M_T = \|T - n_T\|$, it follows from the fact that $M_T = \|T - m_T\| = \min\ \{\|T - \alpha\| : \alpha \in C\} \leq \|T - n_T\| = M_T$, that $n_T = m_T$ (i.e. T is quasi-numeroid) by the uniqueness of the smallest closed disc containing $W(T)$.

Example 3. There is a non-numeroid quasi-numeroid operator.

From Example 1, we have $n_T = m_T$, but $W_T = 1 \neq 2 = M_T$. Therefore the class of numeroid operators is properly contained in the class of quasi-numeroid operators.

3. Spectral set for numeroid operators and normal dilation of numeroid operators

For an operator T on H, a normal operator N acting on a Hilbert space $K \supset H$ is called a normal dilation of T if N satisfies $PNP = TP$ where P is the projection of K onto H. In addition, if N satisfies $PN^nP = T^nP, n = 1, 2, 3, \ldots$, then T is called a strong normal dilation of T. A closed set S in the plane is a spectral set for an operator T if $\sigma(T) \subset S$ and $\|f(T)\| \leq \sup_{\lambda \in S} |f(\lambda)| = \|f\|_S$ for any rational function f with poles off S. Let D be the unit disc of the complex plane and let b_T be the smallest closed disc containing $W(T)$ with its radius W_T and its center n_T.

The following theorem A and Theorem B are fundamental.

Theorem A(Von Newmann [11]) $\{\lambda \in C : |\lambda - \mu| \leq k\}$ is a spectral set for an operator T if and only if $\|T - \mu\| \leq k$.

Theorem B(A. Lebow [7]) Let S be a compact set in the plane and a spectral set for an operator T. Then there exists a strong normal dilation N of T with $\sigma(N) \subset \partial S$ where ∂S denotes the boundary of S

By Theorem A and Theorem B, we shall give a characterization of normaloid operator in the following lemmas:

Lemma 3.1. An operator $T - n_T$ is normaloid if and only if $\omega(T - n_T)D(= W_TD)$ is a spectral set for $T - n_T$.

Proof. if $T - n_T$ is normaloid, we have $\omega(T - n_T) = \gamma(T - n_T) = \|T - n_T\|$. It follows from Theorem A that $\omega(T - n_T)D$ is a spectral set for $T - n_T$. Conversely, if $\omega(T - n_T)D$ is a spectral set for $T - n_T$, then, for a rational function such taht $f(\lambda) = \lambda$, we have that $\|T - n_T\| = \|f(T - n_T)\| \leq \sup_{\lambda \in \omega(T - n_T)D} |f(\lambda)| = \|f\|_{\omega(T-n_T)D}$ and $\|f\|_{\omega(T-n_T)D} \leq \omega(T - n_T) \leq \|T - n_T\|$. Hence $T - n_T$ is normaloid.

Lemma 3.2. $\omega(T - n_T)D$ is a spectral set for an operator $T - n_T$ if and only if there exists a strong normal dilation N_{n_T} of $T - n_T$ with $\|N_{n_T}\| = \omega(T - n_T)$

Proof. Suppose that $\omega(T - n_T)D$ is a spectral set for an operator $T - n_T$. By Theorem B there exists a strong normal dilation N_{n_T} with $\sigma(T_{n_T}) \subset \partial[\omega(T - n_T)D]$. Hence $\|N_{n_T}\| = \omega(T - n_T)$. Conversely, suppose that $T - n_T$ has a strong normal dilation N_{n_T} with $\|N_{n_T}\| = \omega(T - n_T)$ and f is a rational function with poles off $\omega(T - n_T)D$. Since $(f(T - n_T)x, y) = (f(N_{n_T})x, y)$ for $x, y \in H$, we have $\|f(T - n_T)\| \leq \|f(N_{n_T})\|$. Also we have $\sigma(N_{n_T}) \subset \omega(T - n_T)D$ and $\|f(N_{n_T})\| \leq \|f\|_{\sigma(N_{n_T})}$ by the normality of N_{n_T}. It follows that $\omega(T - n_T)D$ is a spectral set for $T - n_T$.

We shall give characterizations of numeroid operators in Theorem 3.3 and Theorem 3.4

Theorem 3.3. An operator T is numeroid if and only if b_T is a spectral set for T.

Proof. If T is a nuneroid operator, $T - n_T$ is normaloid. It follows from Lemma 3.1 that $\omega(T - n_T)D (= W_T D)$ is a spectral set for $T - n_T$, that is, $b_T (= W_T D)$ is a spectral set for T. conversely, if b_T is a spectral set for T, $\omega(T - n_T)D$ is a spectral set for $T - n_T$. It follows from Lemma 3.1that $T - n_T$ is normaloid and so T is a numeroid operator.

Theorem 3.4 An operator T is numeroid if and only if there exists a strong normal dilation N_{n_T} of $T - n_T$ with $\|N_{n_T}\| = W_T$.

Proof. By Theorem 3.3, T is numeroid if and only if b_T is a spectral set of T. By Lemma 3.2 $\omega(T - n_T)D$ is a spectral set of $T - n_T$ if and only if there exists a strong normal dilation N_{n_T} of $T - n_T$ with $\|N_{n_T}\| = \omega(T - n_T) = W_T$.

From Theorem 3.3 and Theorem 3.4, we have that the following statements are equivalent: (1) T is a numeroid operator, (2) b_T is a spectral set for T, and (3) there exists a strong normal dilation N_{n_T} of $T - n_T$ with $\|N_{n_T}\| = W_T$.

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