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# APPROXIMATE MLE IN AN EXPONENTIAL DISTRIBUTION WITH GENERALIZED CENSORING

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## 1.Introduction

A random variable $X$ has an exponential distribution if it has a probability density function(pdf) of the form:

$$(1) \qquad f(x) = \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right),\ x \geq 0,\ \sigma > 0,$$

where $\sigma$ is scale parameter.

Lloyd(1952) described a method of obtaining the best linear unbiased estimators(BLUEs) of the parameters of exponential distribution, using order statistics. Gupta(1952) proposed estimation of the mean and standard deviation of a normal population from a censored sample. The approximate maximum likelihood estimation method was first developed by Balakrishnan(1989a,b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. Lee (2000) obtained the estimation in an exponential distribution with multiply censored sample. Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen(1991).

In this paper, we derive the AMLE of the scale parameter in the one-parameter exponential distribution with pdf(1) based on the generalized censored sample which the $r$ smallest observations and $(n-s-1)$ largest observations are available and $(s-r-1)$ middle observations are censored.

We also obtain the asymptotic variance of the AMLE.

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## 2. Preliminary

Consider one-parameter exponential distribution with density function (1) and cumulative distribution function (cdf)

$$(2) \qquad F(x) = \begin{cases} 1 - \exp(-\frac{x}{\sigma}) & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Let us consider an experiment in which $n$ exponential components are put to test simultaneously at time $x = 0$, and the failure times of there components are recorded.

We will consider the generalized censored sample which include the type-II censored sample. Let

$$(3) \ X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n} \leq X_{s:n} \leq X_{s+1:n} \leq \cdots \leq X_{n:n}$$

be the available censored sample from the exponential distribution with pdf(1), where the middle $(s - r - 1)$ observations are censored.

## 3. Approximate estimation

We shall derive the AMLE of $\sigma$ based on the censored sample in (3). The likelihood function based on the censored sample in (3) is given by

$$(4) \ L = \frac{n!}{(s-r-1)!} \times [F(X_{s:n} : \sigma) - F(X_{r:n} : \sigma]^{s-r-1} \\ \times \prod_{i=1}^{r} f(X_{i:n} : \sigma) \times \prod_{j=s}^{n} f(X_{j:n} : \sigma), \ X_{i:n} \geq 0,$$

which upon denoting $Z_{i:n} = X_{i:n}/\sigma$, can be written as

$$(5) \ L = \frac{n!}{(s-r-1)!} \sigma^{-A} \times [F(Z_{s:n}) - F(Z_{r:n})]^{s-r-1} \\ \times \prod_{i=1}^{r} f(Z_{i:n}) \times \prod_{j=s}^{n} f(Z_{j:n}), \ Z_{i:n} \geq 0$$

where $A = n - s + r + 1$ is the size of censored sample (3), and f(z) and $F(z)$ are the pdf and cdf of the standard exponential distribution, respectively

Now, we will obtain the AMLE of the scale parameter. First, we differentiate the logarithm of the likelihood function (5) for $\sigma$ as follows;

$$(6) \ \frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\{A + (s - r - 1) \\ [\frac{f(Z_{s.n})}{F(Z_{s:n}) - F(Z_{r.n})} \cdot Z_{s:n} - \frac{f(Z_{r:n})}{F(Z_{s:n}) - F(Z_{r:n})} \cdot Z_{r:n}] \\ + \sum_{i=1}^{r} \frac{f'(Z_{i.n})}{f(Z_{i:n})} \cdot Z_{i:n} + \sum_{j=s}^{n} \frac{f'(Z_j)}{f(Z_j)} \cdot Z_j\} = 0$$

Equation (6) does not admit an explicit solution for $\sigma$. But since $\frac{f'(Z_{i:n})}{f(Z_{i:n})} = -1$, we can expand the function

$$H(Z_r, Z_s) = \frac{f(Z_{s:n})}{F(Z_{s:n}) - F(Z_{r:n})}$$

and

$$G(Z_r, Z_s) = \frac{-f(Z_{r:n})}{F(Z_{s:n}) - F(Z_{r:n})}$$

appearing in (6) to Taylor series around the point $(\xi_r, \xi_s)$, where $\xi_i = F^{-1}(p_i) = -\ln(q_i)$ (here, $p_i = \frac{i}{n+1}, q_i = 1 - p_i$) and then approximate it by

$\frac{f(Z_{s:n})}{F(Z_{s:n})-F(Z_{r.n})} \cong \alpha + \beta Z_{s:n} + \gamma Z_{r:n}$

and

$$\text{(7)} \qquad \frac{-f(Z_{r:n})}{F(Z_{s.n})-F(Z_{r:n})} \cong \alpha^* + \beta^* Z_{r:n} + \gamma^* Z_{s:n}$$

$\alpha = \frac{f(\xi_s)}{p_s-p_r}[1 + \xi_s + \frac{f(\xi_s)\cdot\xi_s - f(\xi_r)\cdot\xi_r}{p_s-p_r}]$,
$\alpha^* = \frac{f(\xi_r)}{p_r-p_s}[1 + \xi_r + \frac{f(\xi_r)\cdot\xi_r - f(\xi_s)\cdot\xi_s}{p_r-p_s}]$,
$\beta = -\frac{f(\xi_s)}{(p_s-p_r)^2}[p_s - p_r + f(\xi_s)]$,
$\beta^* = -\frac{f(\xi_r)}{(p_r-p_s)^2}[p_r - p_s + f(\xi_r)]$,
and
$\gamma = \gamma^* = \frac{f(\xi_s)\cdot f(\xi_r)}{(p_s-p_r)^2}$.

Now making use of the approximate expression in (7), we obtain the approximate likelihood equation of (6) as follows ;

$$\text{(8)} \quad \frac{\partial \ln L}{\partial \sigma} \cong \frac{\partial \ln L^*}{\partial \sigma} = -\frac{1}{\sigma}\{A + (s-r-1)[\alpha Z_{s:n} + \alpha^* Z_{r:n} + \beta Z_{s:n}^2 + \beta^* Z_r^2 + (\gamma + \gamma^*) Z_s \cdot Z_{r:n}] - \sum_{i=1}^{r} Z_{i:n} - \sum_{j=s}^{n} Z_{j:n} = 0$$

Since $Z_{i:n} = \frac{X_{i:n}}{\sigma}$, we can derive the AMLE of $\sigma$ as follows;

$$\text{(9)} \qquad \hat{\sigma} = \frac{1}{2A}(-B + \sqrt{B^2 - 4AC})$$

where

$$B = (s-r-1)\alpha X_{s:n} + (s-r-1)\alpha^* X_{r:n} - \sum_{i=1}^{r} X_{i:n} - \sum_{j=s}^{n} X_{j:n}$$

and

$$C = (s-r-1)[\beta X_{s:n}^2 + (\gamma+\gamma^*) X_s X_r + \beta^* X_r^2]$$

These proposed AMLEs admit it explicit estimator. So we can easily estimate the scale parameter by using this estimator.

We simulate the numerical values of $\hat{\sigma}$ by a Monte Carlo simulation method (MSE) for several censoring cases. These values are presented in Table 1.

## 4. Asymptotic properties

Since the AMLEs $\hat{\sigma}$ in (9) is the solutions of the approximate maximum likelihood equations (4), it immediately follows that $\hat{\sigma}$ is asymptotically normally distributed with mean $\sigma$ and variance

$$1/E\{-d^2 \ln L^*/d\sigma^2\}$$

(See Kendall and Stuart (1973)). Now, from equation (4) we can obtain

$$E\left(-\frac{d^2 \ln L^*}{d\sigma^2}\right) = \frac{-(A+2D+3F)}{\sigma^2}, \tag{10}$$

where

$$D = (s-r-1)[\alpha E(Z_{s:n}) + \alpha^* E(Z_{r:n})] - \sum_{i=1}^{r} E(Z_{i:n}) - \sum_{j=s}^{n} E(Z_{j:n})$$

and

$$F = (s-r-1)[\beta E(Z_{s:n}^2) + (\gamma+\gamma^*) E(Z_{s:n}) E(Z_{r:n}) + \beta^* E(Z_{r:n}^2)]$$

From the equation (10), we can compute the asymptotic variance of the AMLE $\hat{\sigma}$ by using the following results (Govindarajulu (1966) and Rao et al.(1991)).

$$E(Z_{i:n}) = 2^{-n}\{\sum_{h=0}^{i-1}\binom{n}{h}S_1(i-h,n-h) - \sum_{h=i}^{n}\binom{n}{h}S_1(h-i+1,h)\}$$

$$\begin{aligned}E(Z_{i:n}^2) =& 2^{-n}\{\sum_{h=0}^{i-1}\binom{n}{h}S_2(i-h,n-h) + S_1^2(i-h,n-h)] \\ &+\sum_{h=i}^{n}\binom{n}{h}[S_2(h-i+1,h) + S_1^2(h-i+1,h)]\},\end{aligned}$$

where $S_k(i,n) = \sum_{l=n-i+1}^{n} 1/l^k, k = 1,2$.

**Table1**. The MSE,s of the AMLE of the scale parameter $\sigma$ in an exponential distribution based on generalized censored sample.

(a) Fulla data

| n | r | s | MSE($\sigma$=1.0) |
|---|---|---|---|
| 10 | 2 | 3 | .09135 |
| 20 | 2 | 3 | .04762 |
| 30 | 2 | 3 | .03236 |
| 40 | 2 | 3 | .02304 |
| 50 | 2 | 3s | .01905 |

(b) $X_{3;n}, X_{4:n}$ are censors.

| n | r | s | MSE($\sigma$=1.0) |
|---|---|---|---|
| 10 | 2 | 5 | .27614 |
| 20 | 2 | 5 | .12603 |
| 30 | 2 | 5 | .07674 |
| 40 | 2 | 5 | .05020 |
| 50 | 2 | 5 | .03742 |

(c) $X_{2;n}, X_{3:n}, X_{4:n}$ are censors.

| n | r | s | MSE($\sigma$=1.0) |
|---|---|---|---|
| 10 | 1 | 5 | .17054 |
| 20 | 1 | 5 | .07861 |

| | | | |
|---|---|---|---|
| 30 | 1 | 5 | .04829 |
| 40 | 1 | 5 | .03324 |
| 50 | 1 | 5 | .02504 |

(d) $X_{3;n}, X_{4:n}, X_{5:n}$ are censors.

| n | r | s | MSE($\sigma$=1.0) |
|---|---|---|---|
| 10 | 2 | 6 | .27979 |
| 20 | 2 | 6 | .12390 |
| 30 | 2 | 6 | .07489 |
| 40 | 2 | 6 | .04870 |
| 50 | 2 | 6 | .03715 |

(e) $X_{4:n}$ is censor.

| n | r | s | MSE($\sigma$=1.0) |
|---|---|---|---|
| 10 | 3 | 5 | .39329 |
| 20 | 3 | 5 | .18753 |
| 30 | 3 | 5 | .11339 |
| 40 | 3 | 5 | .07415 |
| 50 | 3 | 5 | .05416 |

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