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NON COMMUTATIVE TOEPLITZ ALGEBRA OF A CERTAIN SEMIGROUP

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<Abstract>

We analyze the structure of a non-commutative Toeplitz algebra of a semigroup S={0, 3, 4, 6,・・}

어떤 반군에대한 비가환 토에플리츠 대수

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<요약>

반군 S={0, 3, 4, 6, ・・}에대한 비가환 토에플리츠 대수의 구조를 분석하였음.

1. Introduction

The theory of the Toeplitz algebra is one of the motivations of develop the theory of C^*-algebras generated by semigroups of isometries. The C^*-algebra generated by the left regular isometric representation of a left cancellative semigroup is the most appropriate analogue of the Toeplitz algebra among the C^*-algebras generated by isometric representations of semigroups and many of the important results concerning the C^*-algebras generated by semigroups of isometries have been results with the left regular isometric representations of left cancellative semigroups and the C^*-algebras generated by the left regular isometric representations [4,5,7,8, etc].

Though the C^*-algebra which is generated by the left regular representation of a left cancellative semigroup M, a C^*-algebra like the Toeplitz algebra, has been named in the several ways, we prefer to call it the reduced semigroup C^*-algebra and denote it by $C^*_{red}(M)$.

Besides the reduced semigroup C^*-algebra $C^*_{red}(M)$ we are interested in another C^*-algebra generated by the semigroup of isometries, which is called the semigroup C^*-algebra and denoted by $C^*(M)$. The semigroup C^*-algebra $C^*(M)$ is obtained by enveloping all isometric representations of M, and so it is the universal object of the C^*-algebras generated by isometric representations of M (cf.[9]).

Ever since L. A. Coburn proved his well-known theorem [1], which asserts that C^*-algebras generated by a non-unitary isometry on a Hilbert

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space do not depend on the particular choice of the isometry, many authors have contributed to development of generalization of Coburn's result, that is, the uniqueness property [1, 2, 4].

We show that if M is the subsemigroup of the integer group $\mathbb{Z}$ and generates the integer group $\mathbb{Z}$, then $C^*_{red}(M)$ is isomorphic to the Toeplitz algebra. However, we also obtain an example which shows that $C^*_{red}(M)$ is not isomorphic to $C^*(M)$ even when M is a subsemigroup of $\mathbb{Z}$.

2. Generalized Toeplitz Algebra for a semigroup $S = \{0, 3, 4, 6, 7, \dots\}$

In this paper M denotes the countable, left-cancellative semigroup with unit e.

We can give an order on M as follows: if an element x in M is contained in yM for some element $y \in M$, then x and y are comparable and we denote this by $y \leq x$. This relation makes M a pre-ordered semigroup. If the unit of M is the only invertible element of M, the above relation on M becomes a partial order on M. If M is a positive cone of a partially ordered abelian group G and if the condition that $n \in \mathbb{N}$ and $x \in G$ with $nx \in M$ implies that $x \in M$, then a partially ordered abelain group (G, M) is unperforated.

The left regular isometric representation $\mathcal{L}$ of M acts on the Hilbert space $l^2(M)$ by $\mathcal{L}_x(\delta_y) = \delta_{xy}$ for $x, y \in M$ where $\{\delta_x \mid x \in M\}$ is the canonical orthonormal basis of $l^2(M)$. Actually, the reduced semigroup C^*-algebra, which is generated by the left regular isometric representation of M, is the closed linear span of $\{\mathcal{L}_{x_{i_1}} \mathcal{L}^*_{x_{i_2}} \dots \mathcal{L}^*_{x_{i_{2n_i}}} \mathcal{L}_{x_{i_{2n_i+1}}} \mid x_{i_j} \in M\}$. We will consider the reduced group C^*-algebea for a semigroup $S = \{0, 3, 4, 6, 7, \dots\}$. It is well known that the reduced semigroup $C^*_{red}(\mathbb{N})$ and the semigroup $C^*(\mathbb{N})$ for the natural number $\mathbb{N}$ are isomorphic to the Toeplitz algebra. The semigroup S generates the integer group $\mathbb{Z}$ as the natural number semigroup $\mathbb{N}$ generates the integer group $\mathbb{Z}$. But we show that the the reduced semigroup $C^*_{red}(S)$ is isomorphic to the Toeplitz algebra and the semigroup $C^*(S)$ is not isomorphic to the Toeplitz algebra.

Lemma 2.1. *$C^*_{red}(S)$ acts irreducibly on $l^2(S)$.*

Proof. Let T be a bounded operator on $l^2(S)$ commuting with $C^*_{red}(S)$ and $[T_{n,m}]_{n,m\in S}$ be the matrix operator of T with respect to the canonical basis $\{\delta_n \mid n \in S\}$ of $l^2(S)$. We have

$$T_{n,m} = < T(\delta_m), \delta_n > = < T\mathcal{L}^*_n(\delta_m), \delta_0 > = < T(\delta_0), \mathcal{L}^*_m(\delta_n) > .$$

So $T_{n,m} \neq 0$ only when $n \in m + S$ and $m \in n + S$. Since the unit of M is the only invertible element, it follows that $T_{n,m} \neq 0$ only when $m = n$. Furthermore, isometries $\mathcal{L}_n$'s make $T_{n,n}$ scalar operators. So $C^*_{red}(S)$ acts irreducibly on $l^2(S)$. □

Theorem 2.2. *$C^*_{red}(S)$ is isomorphic to the Toeplitz algebra.*

Proof.. We define a compact operator K_0

$$K_0(\delta_n) = \begin{cases} \delta_3, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

And then we define a compact operator F_l such as

$$F_1(\delta_n) = \begin{cases} \delta_6, & n = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}$ be the left regular isometric representation on $l^2(S)$ and put $U = \mathcal{L}^*_3\mathcal{L}_4 + K_0 + F_1$. The compact operator algebra $\mathcal{K}(l^2(S))$ is contained in $C^*_{red}(S)$ because $C^*_{red}(S)$ acts irreducibly on $l^2(S)$, and thus U is contained in $C^*_{red}(S)$. We can see that

$$U(\delta_0) = \mathcal{L}^*_3\mathcal{L}_4(\delta_0) + K_0(\delta_0) + F_1(\delta_0) = \delta_3.$$

Similiarly we have that

$$U(\delta_3) = \delta_4, \quad U(\delta_4) = \delta_6.$$

Furthermore, since $K_0(\delta_n) = 0$ and $F_1(\delta_n) = 0$ for $n > 4$, we have that

$$U(\delta_n) = \delta_{n+1}.$$

Therefore the operator U translates the elements of the canonical orthonormal basis $\{\delta_n \mid n \in S\}$ of $l^2(S)$ to the left, one by one. If we put the C^*-algebra $\mathcal{U}$ of $C^*_{red}(S)$ generated by U, then $\mathcal{U}$ is isomorphic to the Toeplitz algebra.

Eventually, $\mathcal{L}_3$ and $\mathcal{L}_4$ generates $C^*_{red}(S)$, so it is enough to show that $\mathcal{L}_3$ and $\mathcal{L}_4$ can be written as $U +$ {suitable operators in $\mathcal{U}$} in order to say that U generates $C^*_{red}(S)$. So we consider U^3 and U^4. Since the terms of U^3 containing K_0 are removed,

$$U^3 = (\mathcal{L}_3^*\mathcal{L}_4)^3 + \sum (\mathcal{L}_3^*\mathcal{L}_4)^{s_1} F_1^{s_2} (\mathcal{L}_3^*\mathcal{L}_4)^{s_3} \cdots F_1^{s_q}$$

where $s_1 + \cdots + s_q = 3$ and s_i may be zero. In order to make up the gaps of $(\mathcal{L}_3^*\mathcal{L}_4)^3$

we define compact operators M_2 as follows;

$$M_2(\delta_n) = \begin{cases} \mathcal{L}_3(\delta_n), & n = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Due to the compact operators M_2, we have

$$\mathcal{L}_3 = U^3 + M_l - \sum (\mathcal{L}_3^*\mathcal{L}_4)^{s_1} F_1^{s_2} (\mathcal{L}_3^*\mathcal{L}_4)^{s_3} \cdots F_1^{s_q}.$$

Similarly, $\mathcal{L}_4 = U^4 + T$ for a suitable compact operator T. Therefore, U generates $C^*_{red}(S)$ and $C^*_{red}(S)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$.

□

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