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ON CERTAIN CONDITIONS TO BE HARMONIC MAPS BETWEEN QUATERNIONIC KAEHLER MANIFOLDS

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ABSTRACT. Let $(M,g),(N,h)$ be quaternionic Kaehler manifolds of dimensions $4m, 4n$ with almost Hermitian structures $\{I,J,K\},\{F,G,H\}$, and local 1-forms $\{\bar{p},\bar{q},\bar{r}\},\{p,q,r\}$, respectively. Suppose that

$$\phi_*I = F\phi_*, \phi_*J = G\phi_*,$$
$$\bar{p} = \phi^*p, \bar{q} = \phi^*q, \bar{r} = \phi^*r.$$

If there exists an m-dimensional submanifold S of M such that the restriction of ϕ to S is harmonic, then the map $\phi : M \to N$ is harmonic.

1. INTRODUCTION

Let (M,g) and (N,h) be Riemannian manifolds of dimension m and n respectively. For a smooth map $\phi : (M,g) \to (N,h)$ and the *energy density* $e(\phi)$ and the *energy* $E_(\phi)$ of ϕ are respectively defined by

$$e(\phi) := \frac{1}{2}|d\phi|,$$

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$$E(\phi) := \int_M e(\phi)\, dv_g,$$

where $|\cdot|$ denotes the Hilbert-Schmidt norm of the differential $d\phi = \phi_*$ of ϕ which is the differential 1-form with values in the induced vector bundle $f^{-1}TN(TN =$ the tangent bundle of $N)$ over M, and dv_g the volume element of M. A smooth map ϕ is said to be *harmonic* if it is a critical point of the energy functional, that is,

$$\left.\frac{dE(\phi_t)}{dt}\right|_{t=0} = 0$$

for any one-parameter family of maps $\phi_t : M \to N$ with $\phi_0 = \phi$. We denote by ∇ and ${}^N\nabla$ the Levi-Civita connections of M and N respectively. Let $\tilde{\nabla}$ denotes the induced connection on the induced vector bundle $\phi^{-1}TN$ from ${}^N\nabla$ and ϕ. For a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M, we define the *tension field* $\tau(\phi)$ of ϕ by

$$\begin{aligned} \tau(\phi) &:= \sum_{i=1}^m [\tilde{\nabla}_{e_i}(d\phi(e_i)) - d\phi(\nabla_{e_i} e_i)] \\ &= \sum_{i=1}^m (\bar{\nabla}_{e_i} d\phi)(e_i), \end{aligned} \tag{1.1}$$

where $\bar{\nabla}$ is the induced connection on the vector bundle $T^*M \otimes \phi^{-1}TN$.

A smooth map $\phi : M \to N$ is harmonic if and only if the tension field $\tau(\phi) = 0$. If a smooth map $\phi : (M, J) \to (N, J')$ between Kaehler manifolds M, N with almost complex structures J, J', respectively, is holomorphic(i.e., $\phi_* J = J'\phi_*$), then ϕ is a harmonic map. In this situation, our purpose is to find certain conditions for a smooth map between quaternionic Kaehler manifolds to be harmonic. In this article we shall prove the following theorem.

Main Theorem. *Let $(M,g),(N,h)$ be quaternionic Kaehler manifolds of dimensions $4m, 4n$ with almost Hermitian structures $\{I,J,K\}$, $\{F,G,H\}$, and local 1-forms $\{\bar{p},\bar{q},\bar{r}\},\{p,q,r\}$, respectively. Suppose that*

$$\phi_* I = F\phi_*, \phi_* J = G\phi_*, \tag{1.2}$$

$$\bar{p} = \phi^* p, \bar{q} = \phi^* q, \bar{r} = \phi^* r. \tag{1.3}$$

If there exists an m-dimensional submanifold S of M such that the restriction of ϕ to S is harmonic, then the map $\phi : M \to N$ is harmonic.

2. Preliminaries and Proof

Let (M,g) be a $4m$-dimensional quaternionic Kaehler manifold [1,2] with a canonical local basis $\{I,J,K\}$ of a 3-dimensional vector bundle V consisting of tensors of type $(1,1)$ over M . Then there exists a 3-dimensional vector bundle V of tensors of type $(1,1)$ such that in any coordinate neighborhood U of M, there exists a local basis of almost Hermitian structures I,J,K of V satisfying

$$\begin{aligned} &I^2 = J^2 = K^2 = -I (\text{the identity tranformation}),\\ &IJ = -JI = K, JK = -KJ = I, KI = -IK = J. \end{aligned} \tag{2.1}$$

and there exist local 1-forms $\bar{p},\bar{q},\bar{r}$ on U satisfying

$$\begin{aligned} \nabla_X I &= \bar{r}(X)J - \bar{q}(X)K,\\ \nabla_X J &= -\bar{r}(X)I + \bar{p}(X)K,\\ \nabla_X K &= \bar{q}(X)I - \bar{p}(X)J \end{aligned} \tag{2.2}$$

for any vector field X on M.

Let (N,h) be another $4n$-dimensional quaternionic Kaehler manifold with a canonical local basis $\{F,G,H\}$ of a 3-dimensional vector bundle consisting of tensors of type $(1,1)$ over N and local 1-forms p, q, r on a coordinate neighborhood of N satisfying the equations

(2.3)
$$F^2 = G^2 = H^2 = -I(\text{the identity transformation}),$$
$$FG = -GF = H, GH = -HG = F, HF = -FH = G,$$

(2.4)
$$\begin{aligned} {}^N\nabla_Y F &= r(Y)G - q(Y)H, \\ {}^N\nabla_Y G &= -r(Y)F + p(Y)H, \\ {}^N\nabla_Y H &= q(Y)F - p(Y)G \end{aligned}$$

for any vector field Y on N.

Remark. The conditions $\phi_* I = F\phi_*$ and $\phi_* J = G\phi_*$ implies $\phi_* K = H\phi_*$. In fact, $\phi_* K = \phi_* IJ = F\phi_* J = FG\phi_* = H\phi_*$, where we have used (2.1) and (2.3).

To begin with, we prepare the following lemma.

Lemma [3]. *For a smooth map $\phi : M \to N$ between Riemannian manifolds M and N the equation*

$$\tilde{\nabla}_X \phi_* Y - \tilde{\nabla}_Y \phi_* X - \phi_*([X,Y]) = 0$$

holds for any vector fields X, Y on M

Proof of Main Theorem.

Choose a local orthonormal frame field $\{e_i, Ie_i, Je_i, Ke_i : i = 1, 2, \cdots, m\}$ of M such that $\{e_i : i = 1, 2, \cdots, m\}$ is a local orthonormal frame field of S. Then the tension field $\tau(\phi)$ of ϕ is given by

$$
\begin{aligned}
(2.5) \qquad \tau(\phi) = & \sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\phi_* e_i - \phi_* \nabla_{e_i} e_i) \\
& + \sum_{i=1}^{m}(\tilde{\nabla}_{Ie_i}\phi_* Ie_i - \phi_* \nabla_{Ie_i} Ie_i) \\
& + \sum_{i=1}^{m}(\tilde{\nabla}_{Je_i}\phi_* Je_i - \phi_* \nabla_{Je_i} Je_i) \\
& + \sum_{i=1}^{m}(\tilde{\nabla}_{Ke_i}\phi_* Ke_i - \phi_* \nabla_{Ke_i} Ke_i).
\end{aligned}
$$

Using (1.2), (1.3), (2.1), (2.2), Remark and Lemma, we obtain from the second line of (2.5)

(2.6)

$$
\begin{aligned}
& \sum_{i=1}^{m}(\tilde{\nabla}_{Ie_i}\phi_* Ie_i - \phi_* \nabla_{Ie_i} Ie_i) \\
= & \sum_{i=1}^{m}(\tilde{\nabla}_{Ie_i} F\phi_* e_i - \phi_* \nabla_{Ie_i} Ie_i) \\
= & \sum_{i=1}^{m}\{r(\phi_* Ie_i)G(\phi_* e_i) - q(\phi_* Ie_i)H(\phi_* e_i) + F\tilde{\nabla}_{Ie_i}\phi_* e_i - \phi_* \nabla_{Ie_i} Ie_i\} \\
= & \sum_{i=1}^{m}\{r(F\phi_* e_i)G(\phi_* e_i) - q(F\phi_* e_i)H(\phi_* e_i) + F\tilde{\nabla}_{Ie_i}\phi_* e_i - \phi_* \nabla_{Ie_i} Ie_i\} \\
= & \sum_{i=1}^{m}\big[r(F\phi_* e_i)G(\phi_* e_i) - q(F\phi_* e_i)H(\phi_* e_i) \\
& \quad + F\{\tilde{\nabla}_{e_i}\phi_* Ie_i + \phi_*([Ie_i, e_i])\} - \phi_* \nabla_{Ie_i} Ie_i\big] \\
= & \sum_{i=1}^{m}\big[r(F\phi_* e_i)G(\phi_* e_i) - q(F\phi_* e_i)H(\phi_* e_i) \\
& \quad + F\{\tilde{\nabla}_{e_i} F\phi_* e_i + \phi_*(\nabla_{Ie_i} e_i - \nabla_{e_i} Ie_i)\} - \phi_* \nabla_{Ie_i} Ie_i\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \big[r(F\phi_* e_i) G(\phi_* e_i) - q(F\phi_* e_i) H(\phi_* e_i) \\
&\qquad + F\{(\nabla_{e_i} F)\phi_* e_i + F\tilde{\nabla}_{e_i}\phi_* e_i + \phi_* \nabla_{Ie_i} e_i\} \\
&\qquad - F\phi_*\{(\nabla_{e_i} I)e_i + I\nabla_{e_i} e_i\} - \phi_* \nabla_{Ie_i} Ie_i \big] \\
&= \sum_{i=1}^{m} \{ r(F\phi_* e_i) G(\phi_* e_i) - q(F\phi_* e_i) H(\phi_* e_i) \\
&\qquad + r(\phi_* e_i) H(\phi_* e_i) + q(\phi_* e_i) G(\phi_* e_i) \\
&\qquad - \tilde{\nabla}_{e_i}\phi_* e_i + F\phi_*(\nabla_{Ie_i} e_i) \\
&\qquad - \bar{r}(e_i) H(\phi_* e_i) - \bar{q}(e_i) G(\phi_* e_i) \\
&\qquad + \phi_*(\nabla_{e_i} e_i) - \bar{r}(Ie_i) G(\phi_* e_i) \\
&\qquad + \bar{q}(Ie_i) H(\phi_* e_i) - F\phi_*(\nabla_{Ie_i} e_i) \} \\
&= \sum_{i=1}^{m} \{ -\tilde{\nabla}_{e_i}\phi_* e_i + \phi_*(\nabla_{e_i} e_i) \}.
\end{aligned}
$$

Similarly, the third and forth lines of (2.5) yield (2.7) and (2.8), respec tively.

$$
\begin{aligned}
&\sum_{i=1}^{m} (\tilde{\nabla}_{Je_i}\phi_* Je_i - \phi_* \nabla_{Je_i} Je_i) \\
&= \sum_{i=1}^{m} \{ -r(G\phi_* e_i) F(\phi_* e_i) + p(G\phi_* e_i) H(\phi_* e_i) \\
&\qquad + r(\phi_* e_i) H(\phi_* e_i) + p(\phi_* e_i) F(\phi_* e_i) \\
&\qquad - \tilde{\nabla}_{e_i}\phi_* e_i + G\phi_*(\nabla_{Je_i} e_i) \\
&\qquad - \bar{r}(e_i) H(\phi_* e_i) - \bar{p}(e_i) F(\phi_* e_i) \\
&\qquad + \phi_*(\nabla_{e_i} e_i) + \bar{r}(Je_i) F(\phi_* e_i) \\
&\qquad - \bar{p}(Je_i) H(\phi_* e_i) - G\phi_*(\nabla_{Je_i} e_i) \} \\
&= \sum_{i=1}^{m} \{ -\tilde{\nabla}_{e_i}\phi_* e_i + \phi_*(\nabla_{e_i} e_i) \},
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
(2.8) \qquad & \sum_{i=1}^{m}(\tilde{\nabla}_{Ke_i}\phi_* Ke_i - \phi_*\nabla_{Ke_i}Ke_i) \\
&= \sum_{i=1}^{m}\{q(H\phi_* e_i)F(\phi_* e_i) - p(H\phi_* e_i)G(\phi_* e_i) \\
&\quad + q(\phi_* e_i)G(\phi_* e_i) + p(\phi_* e_i)F(\phi_* e_i) \\
&\quad - \tilde{\nabla}_{e_i}\phi_* e_i + H\phi_*(\nabla_{Ke_i}e_i) \\
&\quad - \bar{q}(e_i)G(\phi_* e_i) - \bar{p}(e_i)F(\phi_* e_i) \\
&\quad + \phi_*(\nabla_{e_i}e_i) - \bar{q}(Ke_i)F(\phi_* e_i) \\
&\quad + \bar{p}(Ke_i)G(\phi_* e_i) - H\phi_*(\nabla_{Ke_i}e_i)\} \\
&= \sum_{i=1}^{m}\{-\tilde{\nabla}_{e_i}\phi_* e_i + \phi_*(\nabla_{e_i}e_i)\}.
\end{aligned}
$$

Substituting (2.6), (2.7) and (2.8) into (2.5), we have

$$
\begin{aligned}
\tau(\phi) &= -2\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\phi_* e_i - \phi_*\nabla_{e_i}e_i) \\
&= 0.
\end{aligned}
$$

This completes the proof.□

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